16.1 Line Integrals of Scalar-valued Functions

Def. (here, n is 2 or 3)

$\vec{r}$ has a continuous nonvanishing derivative.

Let $\vec{r} : [a,b] \to \mathbb{V}^n$ be a smooth parameterization of a curve C.
Let the function $f : \mathbb{R}^n \to \mathbb{R}^1$ be continuous on $[a,b]$.
The line integral of f over/along C is

$$\boxed{\int_C f\,ds = \int_{t=a}^{t=b} f(\vec{r}(t))\ \|\vec{r}\,'(t)\|\ dt} \qquad (1)$$

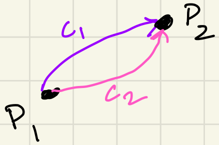
Rmk For $\vec{r}(t) = \langle x_1(t), \dots, x_n(t) \rangle \in \mathbb{V}^n$ ← a vector in $\mathbb{R}^n$
let $\vec{r}_{pt}(t) = (x_1(t), \dots, x_n(t)) \in \mathbb{R}^n$, ← a point in $\mathbb{R}^n$
In (1) above

$f(\vec{r}(t))$ is really $f(\vec{r}_{pt}(t))$.

Rmk Other (better?) terms for Line Integral are: Path Integral and Curve Integral.

Rmk A line integral in independent of paramerization of the curve C. I.e.
if $\vec{r}_1 : [a_1, b_1] \to \mathbb{V}^n$ and $\vec{r}_2 : [a_2, b_2] \to \mathbb{V}^n$ are both smooth paramet. of C
then $\int_{a_1}^{b_1} f(\vec{r}_1(t))\,\|\vec{r}_1\,'(t)\|\,dt = \int_C f\,ds = \int_{a_2}^{b_2} f(\vec{r}_2(t))\,\|\vec{r}_2\,'(t)\|\,dt$.

Warning The value of a line integral btw 2 fixed points can change if we change the path btw the 2 points.



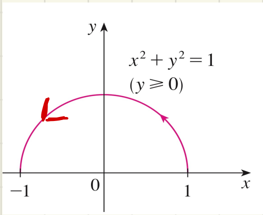
$\int_{C_1} f\,ds$ need not equal $\int_{C_2} f\,ds$.

<u>Recall</u> $\int_C f\,ds = \int_{t=a}^{t=b} f(\vec{r}(t))\, \|\vec{r}\,'(t)\|\,dt$

need to find $\vec{r}(t)$!

<u>Ex 1</u> Evaluate $\int_C (2+x^2y)\,ds$ where C is the upper half of the unit circle $x^2+y^2=1$, orientated counter clock sise.

<u>Soln</u> First find a smooth parameterization $\vec{r}$ of C



$\vec{r}(t) = \langle \cos t, \sin t \rangle$ for $0 \le t \le \pi$.

$x(t) = \cos t$ $\quad y(t) = \sin t$

$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1.$

So

$$\int_C f\,ds = \int_{t=0}^{t=\pi} f(\vec{r}(t))\, \|\vec{r}\,'(t)\|\,dt$$

$$\boxed{\begin{aligned} f(x,y) &= 2+x^2y \\ x(t) &= \cos t \\ y(t) &= \sin t \end{aligned}}$$

$$= \int_0^{\pi} (2+\cos^2 t \sin t)\, \|\vec{r}\,'(t)\|\,dt$$

$$= \int_0^{\pi} (2+\cos^2 t \sin t)\; 1 \; dt$$

TL: let $u = \cos t$ ......

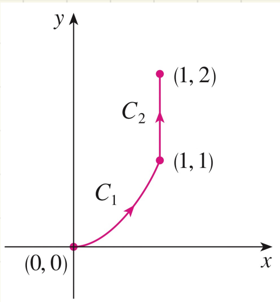
$$= \left[2t - \frac{\cos^3 t}{3} \right]_{t=0}^{t=\pi}$$

$$= \boxed{2\pi + \frac{2}{3}}$$

Recall $\int_C f\,ds = \int_{t=a}^{t=b} f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt$

Ex 2 Evaluate $\int_C 2x\,ds$ where C consists of the curves C_1 of the parabola $y=x^2$ from $(0,0)$ to $(1,1)$ followed by C_2 which is the line segment from $(1,1)$ to $(1,2)$.

Soln



$$\boxed{\begin{gathered} C = C_1 \cup C_2 \\ \Downarrow \\ \int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds \end{gathered}}$$

Given: $f(x,y) = 2x$.

$\boxed{\text{For } C_1}$ $\vec{r}_1(t) = \langle t, t^2 \rangle$ for $0 \le t \le 1$ $\Rightarrow \|\vec{r}_1\,'(t)\| = \sqrt{1^2+(2t)^2} = \sqrt{1+4t^2}$

$$\int_{C_1} f\,ds = \int_{t=0}^{t=1} \underbrace{f(\vec{r}_1(t))}_{f(x,y)=2x \text{ and } x(t)=t}\,\|\vec{r}\,'(t)\|\,dt$$

$$\boxed{\begin{aligned} u &= 1+4t^2 \\ du &= 8t\,dt \end{aligned}}$$

$$= \int_{t=0}^{t=1} 2t\sqrt{1+4t^2}\,dt = \frac{1}{4}\int_{t=0}^{t=1} (1+4t^2)^{1/2}\,(8t)\,dt$$

$$= \frac{1}{4}\cdot\frac{2}{3}(1+4t^2)^{3/2}\Big|_{t=0}^{t=1} = \frac{1}{6}\left[(1+4)^{3/2} - 1^{3/2}\right] = \frac{5\sqrt{5}-1}{6}$$

$\boxed{\text{For } C_2}$ $\vec{r}_2(t) = \langle 1, t \rangle$ for $1 \le t \le 2$ $f(x,y) = 2x$ and $x(t) = 1$

$$\int_{C_2} f\,ds = \int_{t=1}^{t=2} f(\vec{r}_2(t))\,\|\vec{r}_2\,'(t)\|\,dt = \int_{t=1}^{t=2} 2(1)\sqrt{0^2+1^2}\,dt$$

$$= \int_{t=1}^{t=2} 2\,dt = 2t\Big|_{t=1}^{t=2} = 2(2-1) = 2.$$

$\boxed{\text{For } C \overset{\text{i.e.}}{=} C_1 \cup C_2}$

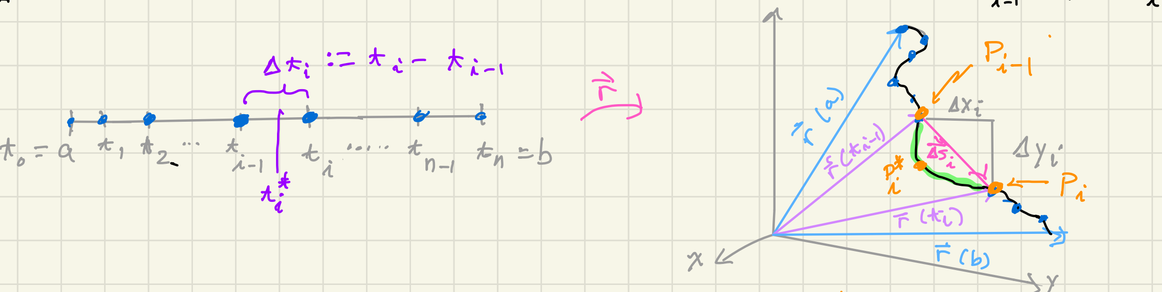
$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds = \frac{5\sqrt{5}-1}{6} + \frac{12}{6} = \boxed{\frac{5\sqrt{5}+11}{6}}$$

What is represented by a line integral of a $\mathbb{R}$-valued function?

Recall the Arc Length lecture (§ 13.3).

- Given a curve C parameterized by $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, $a \le t \le b$.
- Goal: find the arc length AL of C.

Step 1 Partition the interval $[a, b]$. Make a selection t_i^* where $t_{i-1} \le t_i^* \le t_i$.



Let $P_j = \vec{r}_{pt}(t_j)$ and $P_j^* = \vec{r}_{pt}(t_j^*)$.

Step 2 Approx. AL as a sum of "typical elements"

$$AL = \sum_{i=1}^{n} (AL \text{ of part of } C \text{ btw } P_{i-1} \text{ and } P_i) \approx \sum_i \|\Delta \vec{s}_i\|$$

Step 3 Approx. a typical element. We got

$$\Delta \vec{s}_i \approx \underbrace{\vec{r}\,'(t_i^*)}_{\text{a vector}} \underbrace{\Delta t_i}_{\text{a scalar}}$$

$$\|\Delta \vec{s}_i\| \approx \underbrace{\|\vec{r}\,'(t_i^*)\| \, \Delta t_i}_{\text{a scalar}}$$

Step 4 Step 2 and Step 3 $\Rightarrow$ $AL \approx \sum_i \|\vec{r}\,'(t_i^*)\| \Delta t$

Step 5 Take the limit as $\Delta t \to 0$ (so taking more and more t_i's) to get

$$AL = \lim_{\Delta t \to 0} \sum_i \|\Delta \vec{s}_i\| = \lim_{\Delta t \to 0} \sum_i \|\vec{r}\,'(t_i^*)\| \Delta t = \int_{t=a}^{t=b} \|\vec{r}\,'(t)\| \, dt.$$

Take-Aways

Helpful notation:

- $d\vec{s} := \vec{r}\,'(t)\, dt$
- $ds \;\; \frac{\text{think}}{\text{of as}} \;\; "\|d\vec{s}\|" := \|\vec{r}\,'(t)\| \, dt$

Recall we had:

$d\vec{s} \approx \Delta \vec{s}_i \approx \vec{r}\,'(t_i^*) \, \Delta t_i$

$ds \approx \|\Delta \vec{s}_i\| \approx \|\vec{r}\,'(t_i^*)\| \, \Delta t_i$

Ex 3 $\int_C ds \overset{\text{i.e.}}{=} \int_C 1\, ds$ = arc length of C.

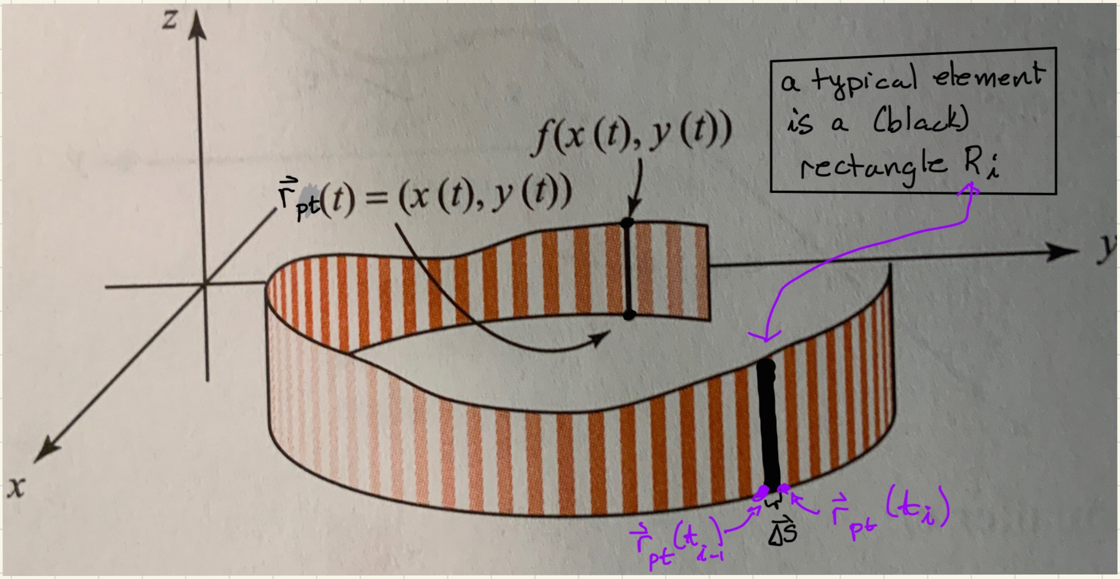
Hint: $\int_C \underset{\uparrow\, f(x,y) = 1}{1}\, ds \overset{\text{def}}{=} \int_a^b \underbrace{f(\vec{r}(t))}_{=1} \|\vec{r}\,'(t)\| \, dt = \int_a^b \|\vec{r}\,'(t)\| \, dt =$

Ex 4 Let curve C have smooth parametrization $\vec{r}(t) = \langle x(t), y(t) \rangle$, for $a \le t \le b$.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous and positive-valued.

Construct a "fence" with base C in xy-plane (= ground) and height $f(x,y)$.

Then proceed similiarly as we did with AL.



- $$\int_C f\,ds \underset{\uparrow\ ds \approx \|\Delta\vec{s}\|}{\approx} \sum_i f(\vec{r}(t_i^*))\,\|\Delta\vec{s}\|$$

- $$\underbrace{f(\vec{r}(t_i^*))}_{\text{(height of } R_i)}\ \underbrace{\|\Delta\vec{s}\|}_{\text{(length of base of } R_i)} \overset{\text{so}}{=} \text{area of } R_i$$

$\Rightarrow \int_C f\,ds \approx \sum_i (\text{area of } R_i) \approx$ the area (of one side) of the whole fence.

Question:

The line integral $\int_C f\,ds$ represents the area (of one side) of the fence.

<u>Rmk</u> The (linear) mass density function δ gives the mass per unit length. The SI (standard international unit) is kg/m.

<u>Ex 5</u> Let a wire lie along curve C having a smooth parameterization $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ for $a \le t \le b$. Let $\delta(t)$ be the mass density of the wire at the point $(x(t), y(t), z(t))$ on C and $\delta : [a,b] \to \mathbb{R}$ be continuous.

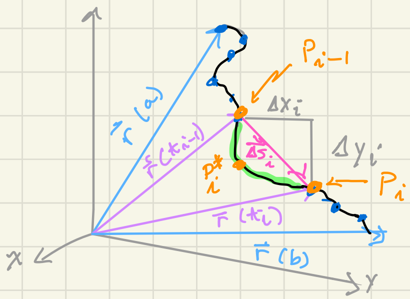
The line integral $\int_C \delta \, ds$ represents total mass of the wire.

<u>Soln.</u>

- $\int_C \delta \, ds \approx \sum_i \delta(\vec{r}(t_i^*)) \|\vec{\Delta s}\| \approx$ mass of whole wire

  $ds \approx \|\vec{\Delta s}\|$

- From AL:



$\Rightarrow \underbrace{\delta(\vec{r}(t_i^*))}_{\text{mass per unit length at } P_i^*} \underbrace{\|\vec{\Delta s}\|}_{\text{length of segment btw } P_{i-1} \text{ and } P_i} \overset{\text{so}}{\approx}$ mass of part of wire btw P_{i-1} and P_i

This finishes 16.1