16.1 Line Integrals of Scalar-valued Functions

Recall: §13.3 Arc Length (AL) from 13.3.1 and 13.3.2

- Given a curve $\mathcal{C}$ parameterized by $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, $a \le t \le b$.
- Goal: Find the arc length AL of $\mathcal{C}$.
- Game Plan: express the Arc Length as an integral w.r.t. t.

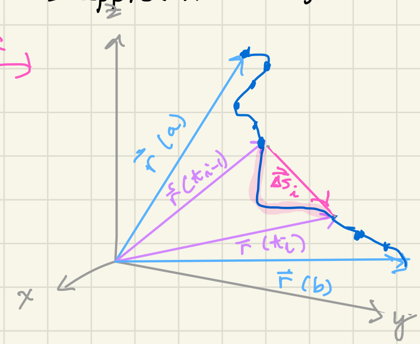
So Want

$$AL = \int_a^b \underbrace{(\text{some mess involving } t)}_{\text{a.k.a. (a function of } t): [a,b] \to \mathbb{R}^1} \, dt$$

(*) An intuitive approach (will use this approach lots).

$t_0 = a \quad t_1 \quad t_2 \quad \cdots \quad t_{i-1} \quad t_i \quad \cdots \quad t_{n-1} \quad t_n = b$

$\Delta t_i := t_i - t_{i-1}$



$$AL \approx \sum_{i=1}^{n} \|\underbrace{\vec{\Delta S_i}}_{\text{Typical element}}\| \approx \sum_{i=1}^{n} (\text{mess}) \, \Delta t_i$$

did some work

$$\vec{\Delta S_i} \overset{\text{def}}{:=} \vec{r}(t_i) - \vec{r}(t_{i-1}) \approx \underbrace{\vec{r}\,'(t_i)}_{\text{a vector}} \, \underbrace{\Delta t_i}_{\text{a scalar}}$$

$$\Rightarrow \|\vec{\Delta S_i}\| \approx \|r'(t_i)\| \, \Delta t_i$$

Now add up all our "typical elements" $\|\Delta S_i\|$

$$AL \approx \sum_{i=1}^{n} \|\vec{\Delta S_i}\| \approx \sum_{i=1}^{n} \|r'(t_i)\| \, \Delta t_i$$

and let $\Delta t \to 0$ (so $n \to \infty$) to get

$$AL = \int_a^b \|r'(t)\| \, dt$$

Helpful notation:

- $d\vec{s} := \vec{r}\,'(t)\, dt$
- ds $\frac{\text{think}}{\text{of as}}$ "$\|d\vec{s}\|$" $:= \underbrace{\|\vec{r}\,'(t)\|\, dt}_{\text{"integrating factor"}}$

Recall we had:

$\leftarrow \vec{\Delta S_i} \approx \vec{r}\,'(t_i) \, \Delta t_i$

$\leftarrow \|\vec{\Delta S_i}\| \approx \|r'(t_i)\| \, \Delta t_i$

End of AL Recall

Line Integrals of Scalar-valued Functions

Goal Given a curve C (where $n = 2$ or 3)

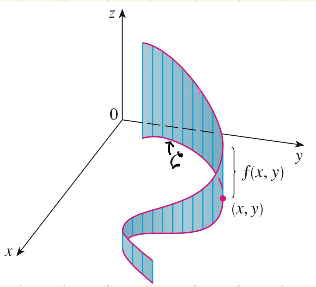
- parametized by $\vec{r} : [a, b] \to \mathbb{R}^n$ ← $\vec{r}(t)$ is $\langle x(t), y(t)\rangle$ or $\langle x(t), y(t), z(t)\rangle$
- and a function $f : \mathbb{R}^n \to \mathbb{R}^1$

Want to integrate f along the curve C, i.e. make sense of

$$\int_C f \, ds ,$$

which we call the <u>line integral of f over C</u>.

<u>Picture</u> for $n=2$ (i.e. $f : \mathbb{R}^2 \to \mathbb{R}$ and $\vec{r} : [a,b] \to \mathbb{R}^2$)



- Note if $f(x,y) = 1$ (for all (x,y))

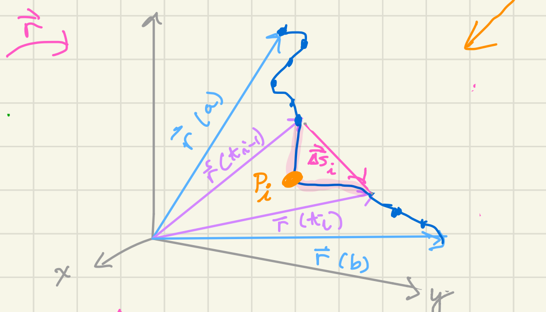
then would want

$$\int_C f \, ds = \text{arclength of } C .$$

Adjust the AL picture by add the $f(\vec{r}(t_i)) := P_i$

$t_0 = a \quad t_1 \quad t_2 \quad \cdots \quad t_{i-1} \quad t_i \quad \cdots \quad t_{n-1} \quad t_n = b$

$\Delta t_i := t_i - t_{i-1}$



So now a "typical element" is

$$f(\vec{r}(t_i)) \, \|\overrightarrow{\Delta s_i}\| \overset{\text{note}}{=} f(\vec{r}(t_i)) \, \|\vec{r}\,'(t_i)\| \, \Delta t_i$$

Add up all the typical elements to get

$$\int_C f ds \approx \sum_i f(\vec{r}(t_i)) \, \|\vec{r}\,'(t_i)\| \, \Delta t_i .$$

Now let $\Delta t \to 0$ to get

$$\int_C f ds = \int_{t=a}^{t=b} f(\vec{r}(t)) \, \|\vec{r}\,'(t)\| \, dt$$

Def. (here, n is 2 or 3) 16.1.3

Let $\vec{r} : [a,b] \to \mathbb{R}^n$ be a smooth parameterization of a curve C.
Let the function $f : \mathbb{R}^n \to \mathbb{R}^1$ be continuous on $[a,b]$.

The line integral of f over/along C is

$$\boxed{\int_C f\,ds = \int_a^b f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt}$$

Ex 1. Application

Any physical interpretation of a line integral $\int_C f(x, y)\,ds$ depends on the physical interpretation of the function f. Suppose that $\rho(x, y)$ represents the linear density at a point (x, y) of a thin wire shaped like a curve C. Then the mass of the part of the wire from P_{i-1} to P_i in Figure 1 is approximately $\rho(x_i^*, y_i^*)\,\Delta s_i$ and so the total mass of the wire is approximately $\Sigma\,\rho(x_i^*, y_i^*)\,\Delta s_i$. By taking more and more points on the curve, we obtain the **mass** m of the wire as the limiting value of these approximations:

$$m = \lim_{n\to\infty} \sum_{i=1}^{n} \rho(x_i^*, y_i^*)\,\Delta s_i = \int_C \rho(x, y)\,ds$$

16.1.4

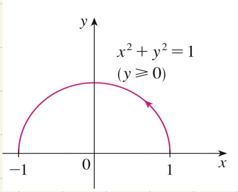
The line integral of f over/along C is

$$\boxed{\int_C f\,ds = \int_a^b f(\vec{r}(t))\ \|\vec{r}\,'(t)\|\ dt}$$

Ex 2 Evaluate $\int_C (2 + x^2 y)\, ds$ where

C is the upper half of the unit circle $x^2 + y^2 = 1$, orientated counter clock sise.

Soln . Find a smooth parameterization of C



$\vec{r}(t) = \langle \cos t, \sin t \rangle$ for $0 \le t \le \pi$

$\Rightarrow \vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2} = \underline{1}.$

So $\displaystyle\int_C f\,ds = \int_{t=0}^{t=\pi} (2 + \cos^2 t \sin t)\ \|r'(t)\|\ dt$

$\displaystyle = \int_0^{\pi} (2 + \underbrace{\cos^2 t \sin t}_{})\, dt =$

TL: let $u = \cos t$ ....

$\displaystyle = \left[2t - \frac{\cos^3 t}{3} \right]_{t=0}^{t=\pi}$

$$= \boxed{2\pi + \frac{2}{3}}$$

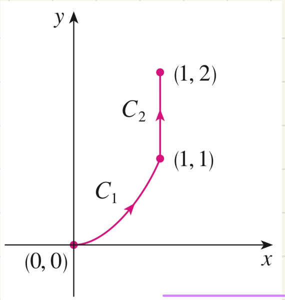
Recall:

The line integral of f over/along C is

$$\boxed{\int_C f\,ds = \int_a^b f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt}$$

Ex 3 Evaluate $\int_C 2x\,ds$ where C consists of the curves C_1 of the parabola $y=x^2$ from $(0,0)$ to $(1,1)$ followed by C_2 which is the line segment from $(1,1)$ to $(1,2)$.

Soln Sketch $C \overset{\text{i.e.}}{=} C_1 \cup C_2$. Here $f(x,y) = 2x$.



1. For C_1: $r_1(t) = \langle t, t^2\rangle$ for $0 \le t \le 1$.

$$\int_{C_1} f\,ds = \int_{t=0}^{t=1} \underbrace{f(\vec{r}_1(t))}_{f(x,y)=2x} \|\vec{r}\,'(t)\|\,dt$$

$$= \int_{t=0}^{t=1} 2t\sqrt{1+(2t)^2}\,dt$$

$u = 1+4t^2$
$du = 8t\,dt$

$$= \int_{t=0}^{t=1} 2t\sqrt{1+4t^2}\,dt = \frac{1}{4}\int_{t=0}^{t=1} (1+4t^2)^{1/2}(8t)\,dt$$

$$= \frac{1}{4}\,\frac{2}{3}\,(1+4t^2)^{3/2}\Big|_{t=0}^{t=1} = \frac{1}{6}\left[(1+4)^{3/2} - 1^{3/2}\right] = \frac{5\sqrt{5}-1}{6}.$$

2. For C_2: $r_2(t) = \langle 1, t\rangle$ for $1 \le t \le 2$.

$$\int_{C_2} f\,ds = \int_{t=1}^{t=2} f(\vec{r}_2(t))\,\|\vec{r}_2\,'(t)\|\,dt = \int_{t=1}^{t=2} 2(1)\sqrt{0^2+1^2}\,dt$$

$$= \int_{t=1}^{t=2} 2\,dt = 2t\Big|_{t=1}^{t=2} = 2(2-1) = 2.$$

3. For all of C: $\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds = \frac{5\sqrt{2}-1}{6} + \frac{12}{6} = \boxed{\frac{5\sqrt{2}+11}{6}}$