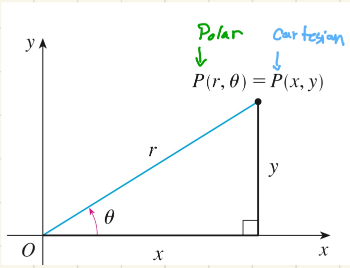
15.7 Triple Integrals in:
- Cylinderical Coords (good for cylinder-ish solid)
- spherical Coords (good for sphere-is Solids)

Cylinderical Coords

(turns 2D polar coords to 3D)

Review

(2D) Polar Coords



$x = r\cos\theta$

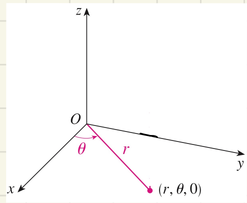
$y = r\sin\theta$

$x^2 + y^2 = r^2$

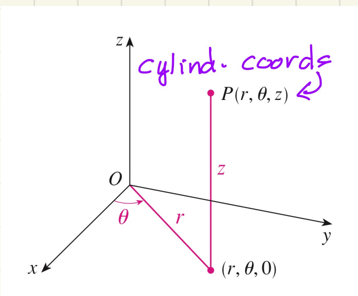
$\theta = \tan^{-1}\frac{y}{x}$

$dA = r\,dr\,d\theta$

Put into 3D



(3D) Cylindrical Coords extend Polar Coords to 3D:



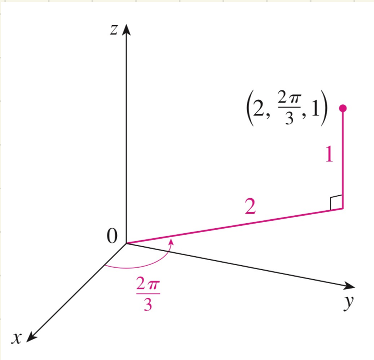
add:

$z = z$

$dV = r\,dz\,dr\,d\theta$

The Cylindrical Coords point $(2, \frac{2\pi}{3}, 1)$

TL: (r, θ, z)



Recall (2D) Polar $\iint_A$

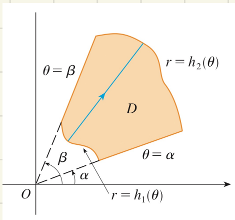
TL: Cartesian Coords $\longrightarrow$ Polar Coords

$dA \longrightarrow dx\,dy \longrightarrow r\,dr\,d\theta$

Thm P Let f be continuous on a polar region

$$D = \{ (r,\theta) : \alpha \le \theta \le \beta,\ h_1(\theta) \le r_2 \le h_2(\theta) \} \qquad (D_*)$$

where h_1 and h_2 are continuous on $[\alpha, \beta]$ and $\beta - \alpha \le 2\pi$.



Then

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

$(*)$

$\iiint$ in Cylindrical Coords

Thm C Let S be a solid in 3D of the form

$$S = \{ (x,y,z) \in \mathbb{R}^3 : (x,y) \in D,\ \ell(x,y) \le z \le u(x,y) \},$$

where
- D is a polar region as in Thm P in (D_*) and
- $z = \ell(x,y)$ and $z = u(x,y)$ are continuous on D.

[TL: So D is projection of S onto the xy-plane]

Let $w = f(x,y,z)$ be continuous on S. Then

$$\iiint_S f(x,y,z)\, dV = \iint_D \left[\int_{z=\ell(x,y)}^{z=u(x,y)} f(x,y,z)\, dz \right] \underbrace{dA}_{dA = r\,dr\,d\theta}$$

$$= \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} \int_{z=\ell(r\cos\theta,\, r\sin\theta)}^{z=u(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

$(*)$

17.5.3

Ex 1

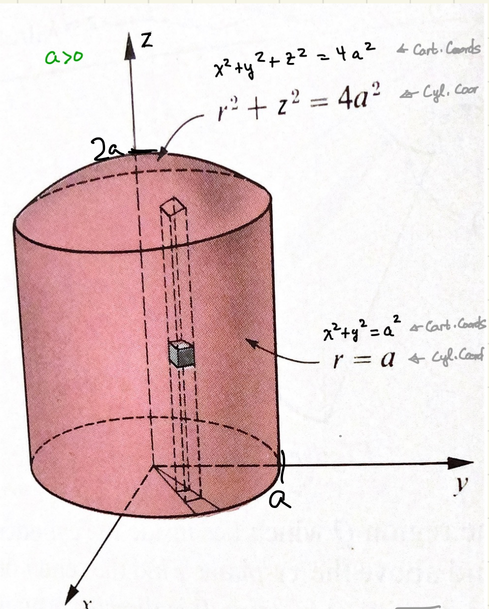
Let S be the (3D) solid that lies: ← Cart. Coords → TL in Cyl. Coords (r,θ,z)

- inside the cylinder $x^2+y^2=a^2$ $\iff r^2=a^2 \iff r=a$
- within the sphere $x^2+y^2+z^2=4a^2$ $\iff r^2+z^2=4a^2$
- above the xy-plane, $\iff z=0$

where $a>0$ is a constant.

Evaluate $\iiint\limits_S z\, dV$.

Soln



Note

- project S onto xy-plane and get

$$D=\{(x,y) : x^2+y^2\le a^2\}$$

i.e. $D=\{(r,\theta) : 0\le r\le a, 0\le\theta\le 2\pi\}$

and

$$S=\{(x,y,z) : (x,y)\in D \text{ and } 0\le z\le \sqrt{4a^2-r^2}\}$$

TL ↑

sphere $r^2+z^2=4a^2$

$z^2=4a^2-r^2$

$z=\pm\sqrt{4a^2-r^2}$

$$\iiint\limits_S z\,dV=\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=a}\left[\int_{z=0}^{z=+\sqrt{4a^2-r^2}} z\,\underbrace{\overset{(*)}{r}\,dz\,dr\,d\theta}_{\text{"}dV\text{"}}\right]$$

$$=\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=a}\left[\frac{rz^2}{2}\Bigg|_{z=0}^{z=\sqrt{4a^2-r^2}}\right]dr\,d\theta$$

You can finish →

$$=\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=a}\frac{r}{2}(4a^2-r^2)\,dr\,d\theta=\frac{1}{2}\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=a}(4a^2r-r^3)\,dr\,d\theta$$

$$=\frac{1}{2}\int_{\theta=0}^{\theta=2\pi}\left(2a^2r^2-\frac{r^4}{4}\right)\Bigg|_{r=0}^{r=a}d\theta=\frac{1}{2}\left(\frac{8}{4}a^4-\frac{a^4}{4}\right)2\pi=\boxed{\frac{7a^4\pi}{4}}$$

Ex 2

Let S be the (3D) solid that lies:

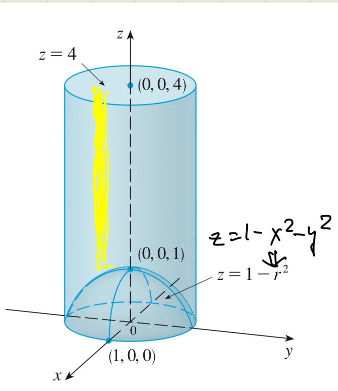
Cart. Coords — in Cyl. Coords (r,θ,z)

- inside the cylinder $x^2+y^2=1$ $\iff r^2=1$
- below the plane $z=4$ $\iff z=4$
- above the paraboloid $z=1-x^2-y^2$ $\iff z=1-r^2$

Evaluate $\iiint_S \sqrt{x^2+y^2}\, dV$.

Soln

Note $\underbrace{\sqrt{x^2+y^2}}_{\text{Cart.}} = \underbrace{\sqrt{r^2}}_{\text{cyl.}} = r$



project S onto the xy-plane and get

$$D = \{(x,y) : x^2+y^2 \le 1\}$$
$$= \{(r,\theta) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$$

and

$$S = \{(x,y,z) : (x,y) \in D \text{ and } \underbrace{1-r^2}_{\text{the paraboloid}} \le z \le 4\},$$

$$\iiint_S \sqrt{x^2+y^2}\, dV = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}\int_{z=1-r^2}^{z=4} \overbrace{r}^{\sqrt{x^2+y^2}}\ \overbrace{r\, dz\, dr\, d\theta}^{dV}$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}\left[\int_{z=1-r^2}^{z=4} r^2\, dz\right] dr\, d\theta$$

you should be able to finish from here... check your soln w/ below.

$$= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}\left[r^2 z \Big|_{z=1-r^2}^{z=4}\right] dr\, d\theta = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} r^2 \overbrace{(4-(1-r^2))}^{3+r^2}\, dr\, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} (3r^2+r^4)\, dr\, d\theta = \int_{\theta=0}^{\theta=2\pi} \left(r^3+\frac{r^5}{5}\right)\Bigg|_{r=0}^{r=1} d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \left(1+\frac{1}{5}\right) d\theta = \left(\frac{6}{5}\right)(2\pi) = \boxed{\frac{12\pi}{5}}$$

Ex 3

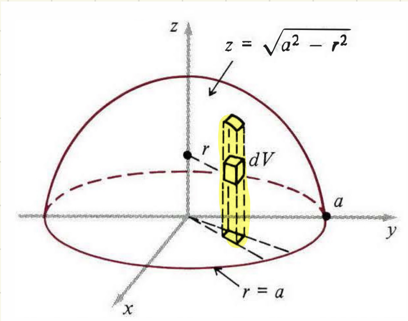
Express the volume of the sphere $x^2+y^2+z^2=a^2$ as a triple integral in cylinderical coordinates. You may use symmetry. Assume $a>0$.

Soln The volume of the sphere = 2 (volume of upper-half sphere).

Let H be the solid upper half sphere $x^2+y^2+z^2=a^2$ and $z\ge 0$.

$\Downarrow$ cyl. coords

$z\ge 0$ and $r^2+z^2=a^2 \Leftrightarrow z=\sqrt{a^2-r^2}$



project H onto the xy-plane and get

$D=\{(x,y) : x^2+y^2\le a^2\}$

$=\{(r,\theta) : 0\le r\le a, \; 0\le\theta\le 2\pi\}$

and

$H=\{(x,y,z) : (x,y)\in D$ and $0\le z\le\sqrt{a^2-r^2}\}$

Volume of sphere = 2 (volume of H, i.e. upper half sphere)

$$= \boxed{2\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=a}\int_{z=0}^{z=\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta}$$

BTW: this integral is nice

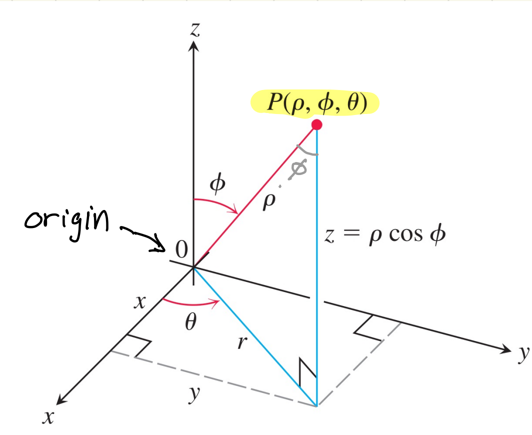
$$=2\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=a}\left[rz\Big|_{z=0}^{\sqrt{a^2-r^2}}\right]dr\,d\theta$$

$$=2\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=a} r(a^2-r^2)^{1/2}\,dr\,d\theta = 2\int_{\theta=0}^{\theta=2\pi} -\frac{1}{3}(a^2-r^2)^{3/2}\Big|_{r=0}^{r=a}\,d\theta$$

$$=2\int_{\theta=0}^{\theta=2\pi}\left(\frac{1}{3}\right)(a^2)^{3/2}\,d\theta = \boxed{\frac{4\pi}{3}a^3}$$

Spherical Coords

Spherical Coords (ρ, θ, ϕ) of a point P in $\mathbb{R}^3$.



$\rho = \|\vec{OP}\| \overset{\text{note}}{\geq} 0$

ϕ = (smallest) angle btw positive z-axis and $\vec{OP}$

so $0 \leq \phi \leq \pi$.

θ = same as the θ in cylinderical coords.

Basic trig gives:

$$r = \rho \sin \phi, \qquad x = r \cos \theta = \rho \sin \phi \cos \theta,$$
$$z = \rho \cos \phi, \qquad y = r \sin \theta = \rho \sin \phi \sin \theta,$$
$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}.$$

Calculus will give

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

Ex1 Express as a triple integral using spherical coords the volume of the sphere, with raduis $a > 0$,

$x^2 + y^2 + z^2 = a^2$. ← cart. coords In sphere. coords it's $\rho^2 = a^2$, i.e. $\rho = a$.

Soln

$$\boxed{V = \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi} \int_{\rho=0}^{\rho=a} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}$$

← give a try before looking below

$$= \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi} \left[\frac{\rho^3}{3} \Big|_{\rho=0}^{\rho=a} \right] \sin\phi \, d\phi \, d\theta$$

$$= \frac{a^3}{3} \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi} \sin\phi \, d\phi \, d\theta = \frac{a^3}{2} \int_0^{2\pi} -\cos\phi \Big|_{\phi=0}^{\phi=\pi} d\theta$$

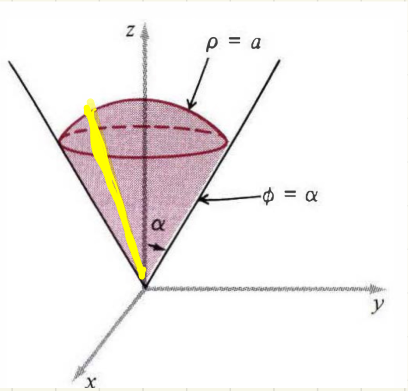
$$= a^3/3 \int_0^{2\pi} 2 \, d\theta = \boxed{\frac{4\pi a^3}{3}}$$

Ex2 : Volume of a filled icecream cone!

Express as a triple integral using spherical coords the volume of the 3D solid bounded by the sphere $\rho = a$ and the cone $\phi = \alpha$

where $a > 0$ and $0 < \alpha < \pi/2$

Soln



$$\boxed{V = \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\alpha} \int_{\rho=0}^{\rho=a} \underbrace{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}_{\text{"}dV\text{"}}}$$

give a try before looking below.

$$= \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\alpha} \left[\frac{\rho^3}{3} \Big|_{\rho=0}^{\rho=a} \right] \sin\phi \, d\phi \, d\theta$$

$$= \frac{a^3}{3} \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\alpha} \sin\phi \, d\phi \, d\theta = \frac{a^3}{3} \int_{\theta=0}^{\theta=2\pi} \left[-\cos\phi \Big|_{\phi=0}^{\phi=\alpha} \right] d\theta$$

$$= \frac{a^3}{3} \int_{\theta=0}^{\theta=2\pi} (-\cos\alpha - -1) \, d\theta = \frac{2\pi a^3}{3} (1 - \cos\alpha)$$

$$\xrightarrow{\alpha \to \pi/2} \frac{2\pi a^3}{3} \left(1 - \cos\frac{\pi}{2}\right) = \frac{2\pi a^3}{3} = \underbrace{\frac{1}{2} \underbrace{\frac{4\pi a^3}{3}}_{\text{volume of sphere}}}_{\text{volume of half-sphere}}$$