§ 15.5 Triple Integrals in Rectangular (i.e. Cartesian) Coords

Def The volume $V(D)$ of a (nice) closed and bounded solid region D in (3D) space. Then

notation

$$V(D) = \iiint_D dV.$$

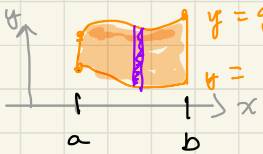
← what does $\iiint$ mean?

Key Idea To make sense of $\iiint_D$, recall $\iint_R$ (§ 15.1)

Setting

(2D) Have a region R in the xy-plane of the form:

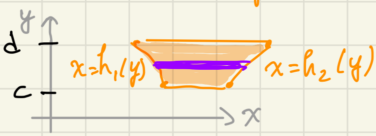
(dy dx) $R = \{ (x,y) \in \mathbb{R}^2 : a \le x \le b \text{ and } g_1(x) \le y \le g_2(x) \}$



with $y = g_1(x)$ and $y = g_2(x)$ continuous on $[a,b]$

or

(dx dy) $R = \{ (x,y) \in \mathbb{R}^2 : c \le y \le d \text{ and } h_1(y) \le x \le h_2(y) \}$



with $x = h_1(y)$ and $x = h_2(y)$ continuous on $[c,d]$

(Now 3D)

Let $D = \{ (x,y,z) \in \mathbb{R}^3 : (x,y) \in R \text{ and } \ell(x,y) \le z \le u(x,y) \}$

with $z = \ell(x,y)$ and $z = u(x,y)$ continuous on R.

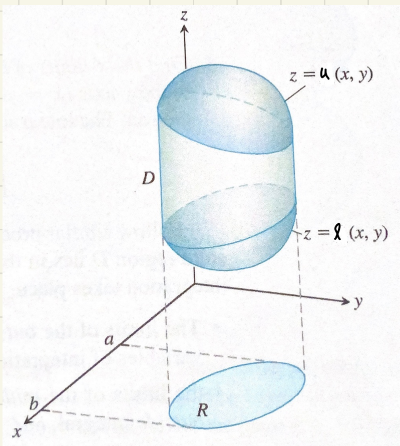


Figure: textbook p 919

Then the volume of D is

$$\boxed{V(D) = \iiint_D dV \overset{\text{def}}{=} \iint_R \left[\int_{z=\ell(x,y)}^{z=u(x,y)} 1 \, dz \right] dA}$$

note reduced $\iiint$ to a $\iint$

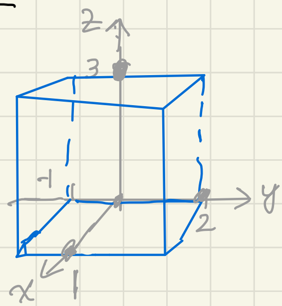
Def. In above setting, if $w = f(x,y,z)$ is continuous on D, then

$$\iiint_D f(x,y,z)\,dV \overset{\text{df}}{=} \iint_R \left[\int_{z=l(x,y)}^{z=u(x,y)} f(x,y,z)\, dz \right] dA,$$

Ex 1. Evaluate $\iiint_B xyz^2\, dV$ where B is the rectangular box

$$B = \{ (x,y,z) \in \mathbb{R}^3 : 0 \le x \le 1,\ -1 \le y \le 2,\ 0 \le z \le 3 \}.$$

Soln



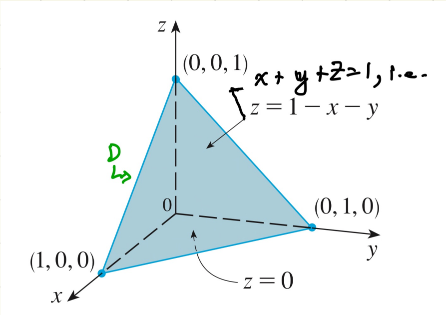
$$\iiint_B xyz^2\, dV =$$

Ex 2. Evaluate $\iiint_D z \, dV$ where D is the solid (so 3D) (tetrahedon) bounded by the 4 planes : 15.5.3

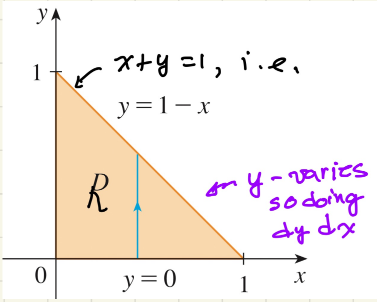
$\underbrace{x=0}_{yz\text{-plane}}$ and $\underbrace{y=0}_{xz\text{-plane}}$ and $\underbrace{z=0}_{xy\text{-plane}}$ and $x+y+z=1$

goes thru 3 pts: $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.

Soln



vary z over R in xy-plane



In R, first we let y vary

$R = \{(x,y) \in \mathbb{R}^2 :$ $\qquad \le y \le$

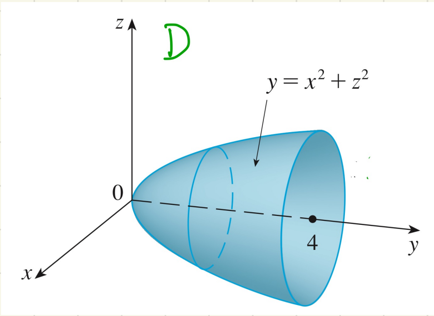
$D = \{(x,y,z) \in \mathbb{R}^3 :$

Ex 3 Evaluate $\iiint_D \sqrt{x^2+z^2}\,dV$ where D is the solid (so 3D) 15.5.4

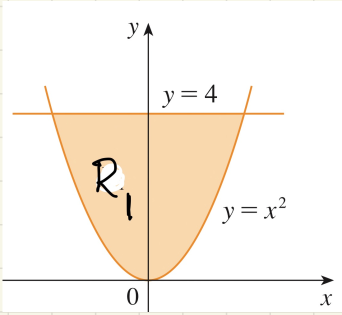
bounded by $\underbrace{y = x^2 + z^2}_{\text{paraboloid}}$ and $\underbrace{y = 4}_{\text{plane}}$.

Soln

- Try 1 - view D as a solid over region R_1 in xy-plane.



Vary z over R_1 in xy-plane



R_1 is as shown above.

$D = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in R_1 \text{ and } \_\_\_\_\_\_ \leq z \leq \_\_\_\_\_\_ \}$

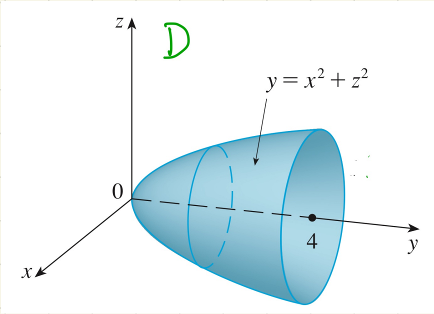
TL: $y = x^2 + z^2 \iff z^2 = y - x^2 \iff z = \pm\sqrt{y - x^2}$

Continued

Ex 3 Evaluate $\iiint_D \sqrt{x^2+z^2}\, dV$ where D is the solid (so 3D) 15.5.5

bounded by $\underbrace{y = x^2 + z^2}_{\text{paraboloid}}$ and $\underbrace{y = 4}_{\text{plane}}$.

- Try 2 - view D as a solid over region R_2 in ____ -plane.



vary

over

R_2

in

__ -plane

write an inequality

$R_2 = \{ (\_\_\_) \in \mathbb{R}^2 : \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \}$.

$D = \{ (x,y,z) \in \mathbb{R}^3 : (\_\_) \in R_2 \text{ and } \_\_\_\_\_\_ \le \_\_ \le \_\_\_\_\_\_ \}$