§ 15.5 Triple Integrals in Rectangular (i.e. Cartesian) Coords

Recall

16.1 Had a curve C in $\mathbb{R}^3$ parameterized by $\vec{r}(t)$.

$\int_C \underbrace{ds}_{\uparrow} =$ arclength of the curve C. $ds = \|\vec{r}\,'(t)\|\,dt$.

TL: "d(arclength)"

15.1-3. Had a region R in $\mathbb{R}^2$.

$\iint_R \underbrace{dA}_{\uparrow} =$ area of the region R. dA can be $dx\,dy$ or $dy\,dx$

TL: "d(Area)"

New 15.5. Have a solid D in $\mathbb{R}^3$. (e.g. D is a domino)

$\iiint_D \underbrace{dV}_{\uparrow} \underset{\uparrow \text{ want}}{=}$ volume of the solid D.

TL: "d(Volume)"

Def. Let D be a (nice) closed and bounded solid region D in (3D) space. The volume of D, denoted $V(D)$, is:

$$\boxed{V(D) = \iiint_D dV}.$$

← what does $\iiint$ mean?

Key Idea To make sense of $\iiint_D dV$, we will reduce $\overbrace{\iiint}^{\text{triple}}$ to a $\overbrace{\iint}^{\text{double}}$ by

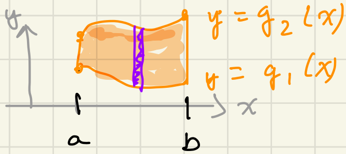
$$\iiint_D dV = \iint_R \left[\int dz\right] dA.$$

Key Idea

Setting

(2D) Have a region R in the xy-plane of the form:

(dy dx) $R = \{ (x,y) \in \mathbb{R}^2 : a \le x \le b \text{ and } g_1(x) \le y \le g_2(x) \}$

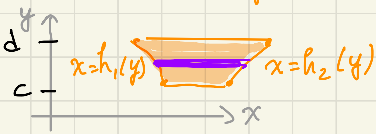


with $y = g_1(x)$ and $y = g_2(x)$ continuous on $[a,b]$

or

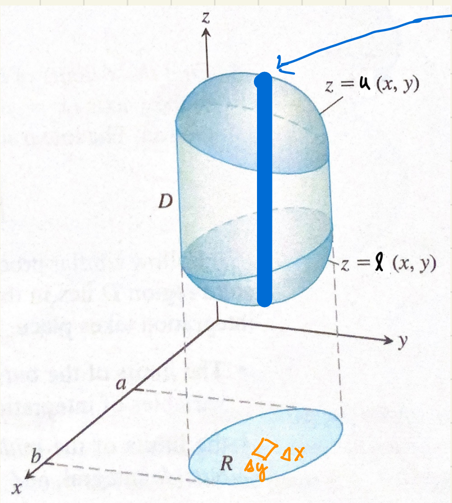
(dx dy) $R = \{ (x,y) \in \mathbb{R}^2 : c \le y \le d \text{ and } h_1(y) \le x \le h_2(y) \}$

with $x = h_1(y)$ and $x = h_2(y)$ continuous on $[c,d]$



(Now 3D)

Let $D = \{ (x,y,z) \in \mathbb{R}^3 : (x,y) \in R \text{ and } \ell(x,y) \le z \le u(x,y) \}$



3D rect. box with base Δx and Δy.

Figure: textbook p 919

with $z = \ell(x,y)$ and $z = u(x,y)$ continuous on R.

Then the volume of D is

$$\boxed{V(D) = \iiint_D dV \overset{\text{df}}{=} \iint_R \underbrace{\left[\int_{z=\ell(x,y)}^{z=u(x,y)} 1 \, dz \right]}_{\text{height of 3D solid blue box}} dA}$$

note reduced $\iiint$ to a $\iint$

Def. In above setting, if $w = f(x,y,z)$ is continuous on D, then

$$\iiint_D f(x,y,z) \, dV \overset{\text{df}}{=} \iint_R \left[\int_{z=\ell(x,y)}^{z=u(x,y)} f(x,y,z) \, dz \right] dA \, .$$

Rmk A Limit of Integration should be a function of the variables remaining to integrate w.r.t. E.g.

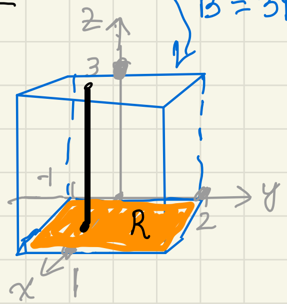
$$\iiint_D z\,dV = \int_{x=c_1}^{x=c_2}\int_{y=g_1(x)}^{y=g_2(x)}\int_{z=f_1(x,y)}^{z=f_2(x,y)} xyz\;dz\,dy\,dx$$

Ex 1. Evaluate $\iiint_B xyz^2\,dV$ where B is the rectangular box

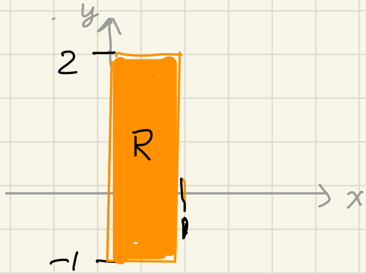
$$B = \{(x,y,z) \in \mathbb{R}^3 : 0 \le x \le 1,\ -1 \le y \le 2,\ 0 \le z \le 3\}.$$

Soln

B = 3D rectangular box



vary z over R in xy-plane



$$\iiint_B xyz^2\,dV = \iint_{\substack{\text{above R}\\ \text{in xy-plane}}}\left[\int_{z=0}^{z=3} xyz^2\,dz\right]dA = \int_{x=0}^{x=1}\left[\int_{y=-1}^{y=2}\left[\int_{z=0}^{z=3} xyz^2\,dz\right]dy\right]dx$$

Fubini

$$\int_{z=0}^{z=3}\left[\int_{y=-1}^{y=2}\left[\int_{x=0}^{x=1} xyz^2\,dx\right]dy\right]dz$$

$$= \int_{z=0}^{z=3}\int_{y=-1}^{y=2}\boxed{\frac{x^2yz^2}{2}\Bigg|_{x=0}^{x=1}}\,dy\,dz = \int_{z=0}^{z=3}\int_{y=-1}^{y=2}\frac{yz^2}{2}\,dy\,dz \overset{\text{will get, doing as in previous §!}}{=} \boxed{\frac{27}{4}}$$

$$\|$$

Integration Details

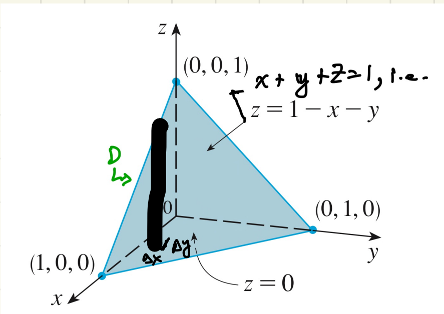
$$= \int_{z=0}^{z=3}\int_{y=-1}^{y=2}\frac{yz^2}{2}\,dy\,dz = \int_{z=0}^{z=3}\frac{y^2z^2}{4}\Bigg|_{y=-1}^{y=2}dz = \int_{z=0}^{z=3}\frac{3z^2}{4}\,dz = \frac{z^3}{4}\Bigg|_{z=0}^{z=3}$$

Ex 2. Evaluate $\iiint_D z\, dV$ where D is the (3D) solid (tetrahedon) bounded by the 4 planes: 15.5.4

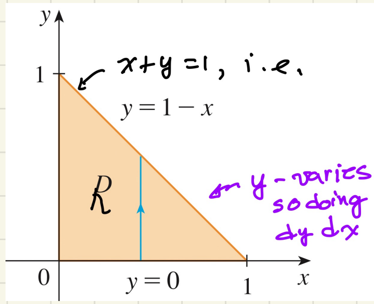
$\underbrace{x=0}_{yz\text{-plane}}$ and $\underbrace{y=0}_{xz\text{-plane}}$ and $\underbrace{z=0}_{xy\text{-plane}}$ and $x+y+z=1$

goes thru 3 pts: $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.

Soln



vary z over R in xy-plane



In R, first we let y vary

$R = \{(x,y) \in \mathbb{R}^2 : \underline{0} \le y \le \underline{1-x}$ and $\underline{0} \le x \le \underline{1}\}$.

$D = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in R$ and $\underline{0} \le z \le \underline{1-x-y}\}$.

$$\iiint_D z\, dV = \iint_{R \text{ (as above)}} \left[\int_{z=0}^{z=1-x-y} z\, dz\right] dA = \iint_R \left[\frac{z^2}{2}\Big|_{z=0}^{z=1-x-y}\right] dA$$

$$= \frac{1}{2}\iint_R (1-x-y)^2\, dA = \frac{1}{2}\int_{x=0}^{x=1}\int_{y=0}^{y=1-x} (1-x-y)^2\, dy\, dx \overset{\text{do as prev. Ex to get}}{=} \boxed{\frac{1}{24}}$$

Integration Details

$$= \frac{1}{2}\int_{x=0}^{x=1} -\frac{(1-x-y)^3}{3}\Big|_{y=0}^{y=1-x} dx = \frac{1}{6}\int_{x=0}^{x=1} [0 - (1-x)^3]\, dx$$

$$= \frac{1}{6}\left(\frac{-(1-x)^4}{4}\right)\Bigg|_{x=0}^{x=1} = \frac{1}{6}\left(0 - \frac{-1}{4}\right) = \frac{1}{6(4)} = \frac{1}{24}.$$

Ex 3 Express the triple integral in previous example as a iterated integral

Soln. $$\iiint_D z\, dV = \int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\int_{z=0}^{z=1-x-y} z\, dz\, dy\, dx$$

15.5.5

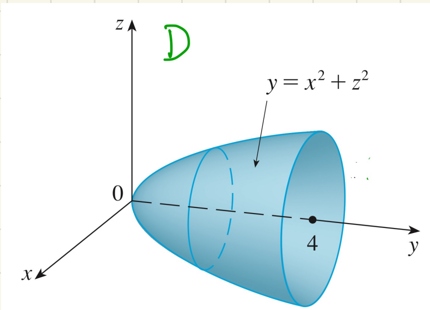
Ex 4 Let D be the solid (so 3D) bounded by

$$\underbrace{y = x^2 + z^2}_{\text{paraboloid}} \quad \text{and} \quad \underbrace{y = 4}_{\text{plane.}}$$

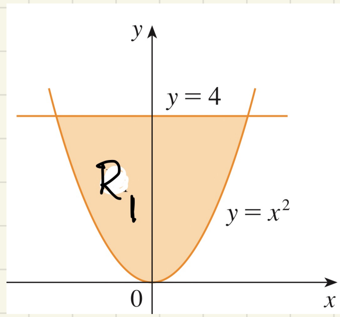
Express $\iiint_D \sqrt{x^2+z^2}\, dV$ as a double integral of a region R_1 in the xy-plane.

Soln Want $\iiint_D \sqrt{x^2+z^2}\, dV = \iint_{R_1} \left[\int_{z=l(x,y)}^{z=u(x,y)} \sqrt{x^2+z^2}\, dz \right] dA.$

So view D as a solid above-&-below a region R_1 in xy-plane.



Vary z over R_1 in xy-plane



$R_1 = \{ (x,y) \in \mathbb{R}^2 : \underline{-2} \le x \le \underline{2} \text{ and } \underline{x^2} \le y \le \underline{4} \}.$

$D = \{ (x,y,z) \in \mathbb{R}^3 : (x,y) \in R_1 \text{ and } \underline{-\sqrt{y-x^2}} \le z \le \underline{+\sqrt{y-x^2}} \}$

TL: $y = x^2 + z^2 \iff z^2 = y - x^2 \iff z = \pm\sqrt{y-x^2}$

$$\boxed{\iiint_D \sqrt{x^2+z^2}\, dV = \iint_{R_1} \left[\int_{z=-\sqrt{y-x^2}}^{z=+\sqrt{y-x^2}} \sqrt{x^2+z^2}\, dz \right] dA.} \quad (*)$$

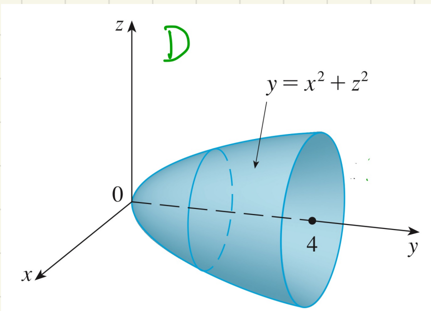
☹ $\int \sqrt{x^2+z^2}\, dz$ is not fun ... and is long ... trig subst.

$u = \sqrt{x}\tan z \rightsquigarrow \int x \sec^3 z\, dz \overset{\text{rewrite}}{=} \int x(1+\tan^2 z)\sec z\, dz$

☹ Similarly with $\int \sqrt{x^2+z^2}\, dx$

Ex 5 Evaluate $\iiint_D \sqrt{x^2+z^2}\, dV$ where D is the solid bounded by $\underbrace{y = x^2+z^2}_{\text{paraboloid}}$ and $\underbrace{y=4}_{\text{plane}}$.

Soln In previous Ex., we saw computing the integral was hard if we integrated first w.r.t. the variable z or x.
So let integrate first w.r.t. the variable y.
So view D as a solid over region R_2 in ____ -plane.



Vary y over R_2 in xz-plane

write an inequality

$R_2 = \{ (x,z) \in \mathbb{R}^2 : x^2+z^2 \le 4 \}$.

$D = \{ (x,y,z) \in \mathbb{R}^3 : (x,z) \in R_2 \text{ and } x^2+z^2 \le y \le 4 \}$

$$\iiint_D \sqrt{x^2+z^2}\, dV = \iint_{R_2} \left[\int_{y=x^2+z^2}^{y=4} \sqrt{x^2+z^2}\, dy \right] dA$$

$$= \iint_{R_2} \left[(4-x^2-z^2)\sqrt{x^2+z^2} \right] dA$$

TL: To calc. the $\iint_{R_2}$, we can use Rectangular or Polar Coords. ← easier
so $x = r\cos\theta$, $z = r\sin\theta$, $x^2+z^2=r^2$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \left[(4-r^2)\sqrt{r^2} \right] \overbrace{r\,dr\,d\theta}^{dA} = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} (4r^2 - r^4)\, dr\, d\theta = \boxed{\frac{128\pi}{15}}$$

will get

Integration Details: $= \int_{\theta=0}^{\theta=2\pi} \left[\left(\frac{4r^3}{3} - \frac{r^5}{5} \right) \Big|_{r=0}^{r=2} \right] d\theta = \underbrace{(2\pi)}_{d\theta}\, 2^3 \left[\frac{4}{3} - \frac{2^2}{5} \right] = 2^6\pi\left(\frac{1}{3} - \frac{1}{5}\right) = 64\pi\left(\frac{5-3}{15}\right) =$