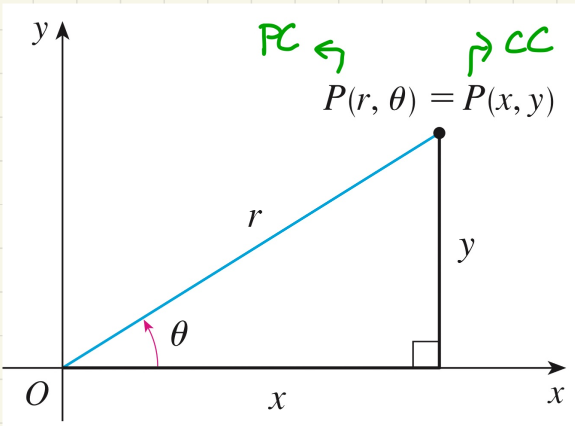
15.4 . Double Integrals in Polar Coordinates.

Recall Polar Coordinates (r,θ) from Calc. II.
If needed, see handout posted on Handout page (under 15.4)

Our old trustly friend, Cartesian coordinates, are handy when dealing with *boxy* objects.
Our new friend, polar coordinates, are handy when dealing with *windy/circular* objects.

let's abbreviate:

Cartesian coordinates by CC (x,y) and *polar coordinates* by PC (r,θ).

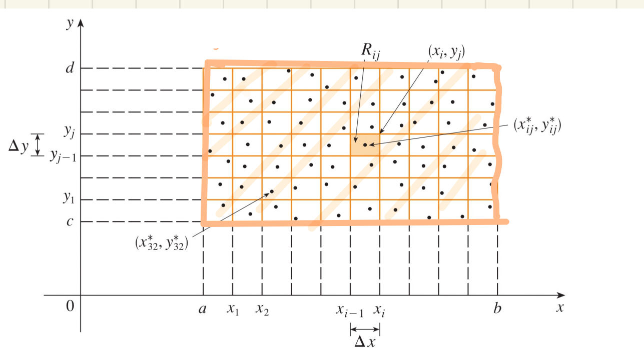


relations btw. CC and PC

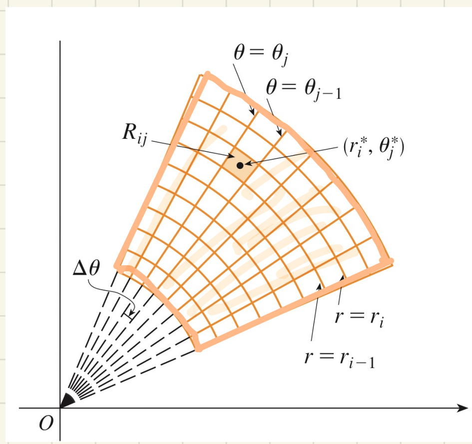
$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

$$x^2 + y^2 = r^2 .$$

(*) When computing double integrals over a region R, we partition R
- (for CC in 15.1-15.3) into small rectangles R_{ij}
- (for PC in 15.4) into small "polar rectangles" R_{ij}



Cartesian Coords

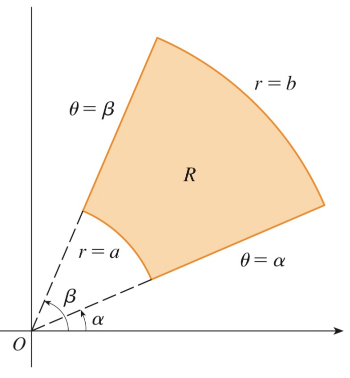


Polar Coords

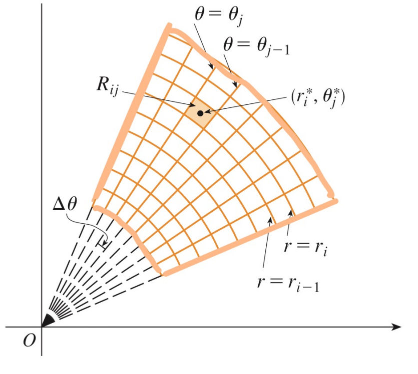
polar rectangle

$$R = \{(r, \theta) \mid a \leqslant r \leqslant b, \alpha \leqslant \theta \leqslant \beta\}$$

15.4.2



Polar rectangle



Dividing R into polar subrectangles

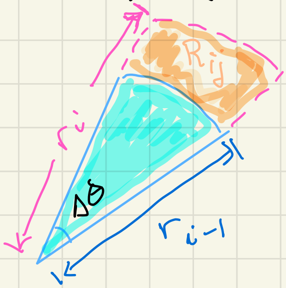
The "center" of R_{ij} is (r_i^*, θ_j^*) where

$$r_i^* = \tfrac{1}{2}(r_{i-1} + r_i)$$

$$\theta_j^* = \tfrac{1}{2}(\theta_{j-1} + \theta_j)$$

Area of R_{ij} = (area of big sector w/ $\measuredangle = \Delta\theta$ and raduis $= r_i$)

− (area of little sector w/ $\measuredangle = \Delta\theta$ and $r = r_{i-1}$)



$$= \tfrac{1}{2} r_i^2 \Delta\theta - \tfrac{1}{2} r_{i-1}^2 \Delta\theta = \tfrac{1}{2}(r_i^2 - r_{i-1}^2)\Delta\theta$$

$$= \tfrac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta = \boxed{r_i^* \Delta r \Delta\theta = \text{area of } R_{ij}}$$

So

$$\iint_R f(x,y)\, dA \approx \sum_j \sum_i f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*) \text{ (area of } R_{ij})$$

$$\approx \sum_j \sum_i f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\, r_i^* \Delta r \Delta\theta$$

As usual, take the limit as $\Delta r \to 0$ and $\Delta\theta \to 0$ to get

$$\iint_R f(x,y)\, dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=a}^{r=b} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

(*) note.

Thm CHANGE TO POLAR COORDINATES IN A DOUBLE INTEGRAL

If $f : R \to \mathbb{R}^1$ is continuous on the polar rectangle

$$R = \{ (r,\theta) \in \mathbb{R}^2 : 0 \le a \le r \le b \text{ and } \alpha \le \theta \le \beta \}$$

where $\beta - \alpha \le 2\pi$, then

$$\iint_R f(x,y)\, dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=a}^{r=b} f(r\cos\theta, r\sin\theta)\, \underline{r}\, dr\, d\theta$$

$(*)$

Recall $\iint_R 1\, dA$ = (total) area of R.

Thm Let $\beta - \alpha \le 2\pi$. Let h_1 and h_2 be continuous on $[\alpha, \beta]$.

If f is continuous on a polar region of the form

$$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta,\ h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

then $$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

$(*)$

TL. Cartesian Coords $\longrightarrow$ Polar Coords

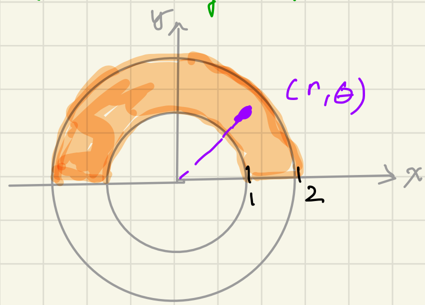
$dA \longrightarrow dx\, dy \longrightarrow r\, dr\, d\theta$

Ex 1 Evaluate

$$\iint_R (3x + 4y^2)\, dA$$

where R is the region in the upper half plane bounded by $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Soln. Sketch R.



So $R = \{(r, \theta) \in \mathbb{R}^2 : 1 \le r \le 2 \text{ and } 0 \le \theta \le \pi\}$.

$$\iint_R (3x+4y^2)\, dA = \int_{\theta=0}^{\theta=\pi} \int_{r=1}^{r=2} (3r\cos\theta + 4r^2\sin^2\theta)\, r\, dr\, d\theta$$

$$= \int_{\theta=0}^{\theta=\pi} \left[\int_{r=1}^{r=2} (3r^2\cos\theta + 4r^3\sin^2\theta)\, dr \right] d\theta$$

$$= \int_{\theta=0}^{\theta=\pi} \left[(r^3\cos\theta + r^4\sin^2\theta) \Big|_{r=1}^{r=2} \right] d\theta$$

$$= \int_{\theta=0}^{\theta=\pi} \left[(8\cos\theta + 16\sin^2\theta) - (\cos\theta + \sin^2\theta) \right] d\theta$$

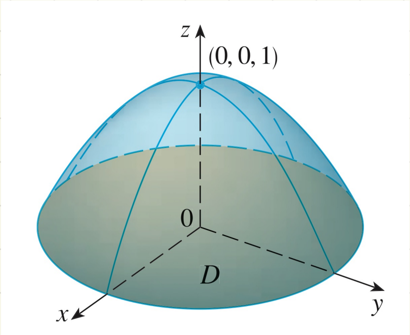
$$= \int_{\theta=0}^{\theta=\pi} \left[7\cos\theta + 15\sin^2\theta \right] d\theta = \int_{\theta=0}^{\theta=\pi} \left[7\cos\theta + \frac{15}{2}(1-\cos 2\theta) \right] d\theta$$

$$= \left(7\sin\theta + \frac{15}{2}\theta - \frac{15}{2}\cdot\frac{1}{2}\sin(2\theta) \right) \Big|_{\theta=0}^{\theta=\pi} \overset{\text{Alg}}{=} \boxed{\frac{15\pi}{2}}$$

15.5 5

Ex 2 Find the volume V of the solid bounded by the plane $z=0$ and the paraboloid $z=1-x^2-y^2$.

Soln



When $z=0$ (i.e. the xy-plane), the paraboloid $z=1-x^2-y^2$ becomes

$$0=1-x^2-y^2 \Rightarrow x^2+y^2=1$$

So D is

$$D=\{(r,\theta)\in\mathbb{R}^2 : 0\le r\le 1,\ 0\le\theta\le 2\pi\}$$

So

$$V=\iint_D (1-x^2-y^2)\,dA \overset{(A)}{=} \iint_D (1-(x^2+y^2))\,dA$$

$x=r\cos\theta$
$y=r\sin\theta$

$$=\int_{\theta=0}^{\theta=2\pi}\left[\int_{r=0}^{r=1}(1-(r^2\cos^2\theta+r^2\sin^2\theta))\,r\,dr\right]d\theta$$

$$=\int_{\theta=0}^{\theta=2\pi}\left[\int_{r=0}^{r=1}(1-r^2)\,r\,dr\right]d\theta$$

$$=\int_{\theta=0}^{\theta=2\pi}\left[\int_{r=0}^{r=1}(r-r^3)\,dr\right]d\theta$$

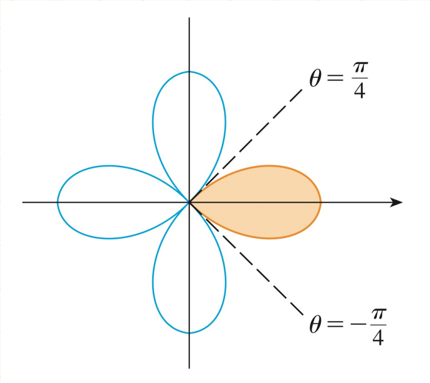
$$=\int_{\theta=0}^{\theta=2\pi}\left[\left(\frac{r^2}{2}-\frac{r^4}{4}\right)\Bigg|_{r=0}^{r=1}\right]d\theta=\int_{\theta=0}^{\theta=2\pi}\frac{1}{4}\,d\theta$$

$$=\frac{1}{4}\theta\Big|_{\theta=0}^{\theta=2\pi}=\frac{2\pi}{4}=\boxed{\frac{\pi}{2}}$$

Ex 3 Find the area A enclosed by one loop of the four leaved rose 15.5.6

$$r = \cos(2\theta)$$

Soln Sketch $r = \cos(2\theta)$



so

$$D = \{(r,\theta) \in \mathbb{R}^2 : -\pi/4 \le \theta \le \pi/4 \text{ and } 0 \le r \le \cos(2\theta)\}.$$

so

$$A = \iint_R 1 \, dA$$

$$= \int_{\theta=-\pi/4}^{\theta=\pi/4} \left[\int_{r=0}^{r=\cos(2\theta)} 1 \quad r\,dr \right] d\theta$$

$$= \int_{\theta=-\pi/4}^{\theta=\pi/4} \left[\frac{r^2}{2} \Big|_{r=0}^{r=\cos(2\theta)} \right] d\theta$$

$$= \frac{1}{2} \int_{\theta=-\pi/4}^{\theta=\pi/4} \cos^2(2\theta) \, d\theta = \frac{1}{2} \int_{\theta=-\pi/4}^{\theta=\pi/4} \frac{1}{2}(1+\cos 4\theta) d\theta$$

$$= \frac{1}{4} \left(\theta + \frac{1}{4} \sin 4\theta \right) \Big|_{\theta=-\pi/4}^{\theta=\pi/4} \quad \text{A} \quad \boxed{\frac{\pi}{8}}$$