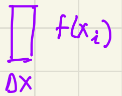
15.1 – 1.3 : Double (Iterated) Integals over Region R in xy-plane

Ex 1 Desmos Demonstration 15.1.1. Review of $\int_a^b f(x)\,dx$ (as a limit of Riemann Sums)

$\downarrow$

Take Away $\int_a^b f(x)\,dx \approx \sum_i f(x_i)\Delta x$

a typical element = rectangle  and $f(x_i)\Delta x$ is the area of typical elt.

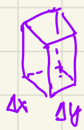
$\underbrace{\sum_i f(x_i)\Delta x}_{\downarrow} = \sum_i (\underline{\text{area}}$ of a typical element$) \approx \underline{\text{area}}$ between $[a,b]$ and $y=f(x)$

Take the limit as $\Delta x \to 0$ to get

$\int_a^b f(x)\,dx = \underline{\text{area}}$ between $[a,b]$ and $y=f(x)$

Goal in §15.1–15.3 To calculate, and make sense of, a double (interated) integral $\boxed{\iint_R f(x,y)\,dA}$ where R is a region in the xy-plane. ⟨A is for Area⟩

Ex 2 Desmos 15.1.2. Double Integral : What does it represent?

Take Away a typical element = rectangluar box  height $= f(x_i, y_j)$ area of base $= (\Delta x)(\Delta y)$

$$\iint_R f(x,y)\,\underbrace{dA}_{\text{change in area } A} \approx \sum_j \sum_i f(x_i, y_i)\underbrace{(\Delta x)(\Delta y)}$$

$$\sum_j \sum_i \underbrace{f(x_i, y_i)}_{\text{(height)}}\underbrace{(\Delta x_i)(\Delta y_i)}_{\text{(area of base)}} = \sum_j \sum_i (\underline{\text{volume}} \text{ of a typical element / rectangular box})$$

of rectangular box

$\downarrow\downarrow\downarrow$ Take the limit as $\Delta x \to 0$ and $\Delta y \to 0$ to get $\downarrow\downarrow\downarrow$

$\iint_R f(x,y)\,dA$ = the $\underline{\text{volume}}$ of the (3D) solid between between the 2D region R and the surface $z = f(x,y)$.

Rmk Revisit Desmos 15.1.2. When adding together volume of the rectangular boxes (i.e., the "typical elements"), what order to use?

Thm. Fubini's Thm for R = rectangle [circa 1908]

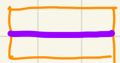
Let $z = f(x,y)$ with $f : R \to \mathbb{R}$ be continuous where

$R = [a,b] \times [c,d] \overset{\text{i.e.}}{=} \{(x,y) \in \mathbb{R}^2 \; ; a \le x \le b$ and $c \le y \le d\}$. Then

$$\iint_R f(x,y)\, dA = \int_{y=c}^{y=d} \underbrace{\left[\int_{x=a}^{x=b} f(x,y)\, dx \right]}_{\text{do 1st for "fixed"}} dy = \int_{x=a}^{x=b} \underbrace{\left[\int_{y=c}^{y=d} f(x,y)\, dy \right]}_{\text{do 1st for "fixed"}} dx$$

TL. For region R

1st
x varies
y fixed



1st
y varies
x fixed

in short
$$\int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x,y)\, dx\, dy = \int_{x=a}^{x=b} \int_{y=c}^{y=d} f(x,y)\, dy\, dx$$

Remark Fubini says we can "reverse the order of integration"
Sometimes one ordering is much easier to integrate than the other order.
TL: thinking before start calculating might make life easier.

Rmk $\iint_R f(x,y)\, dA$ represents the volume of the (3D) solid between the 2D region R and the surface $z = f(x,y)$.

Being more precise, it is a signed volume where
↳ (+/−)

$f(x,y) > 0$ (i.e. $f(x,y)$ is above the xy-plane) gives ⊕ volume while
$f(x,y) < 0$ (i.e. $f(x,y)$ is below the xy-plane) gives ⊖ volume.

Loosely speaking,

$$\iint_R f(x,y)\, dA = \begin{pmatrix} \text{positive (true/actual)} \\ \text{volume of part} \\ \text{of the solid where} \\ f(x,y) > 0 \\ \text{(so } f(x,y) \text{ is above xy-plane)} \end{pmatrix} - \begin{pmatrix} \text{positive (true/actual)} \\ \text{volume of part} \\ \text{of the solid where} \\ f(x,y) < 0 \\ \text{(so } f(x,y) \text{ below xy-plane} \end{pmatrix}$$

Ex 3. Evaluate $\int_0^3 \int_1^2 x^2 y \, dy \, dx$ better to write $\int_{x=0}^{x=3} \left[\int_{y=1}^{y=2} x^2 y \, dy \right] dx$

Soln

$$\int_{x=0}^{x=3} \left[\int_{y=1}^{y=2} x^2 y \, dy \right] dx$$

$$= \int_{x=0}^{x=3} \left[\frac{x^2 y^2}{2} \Big|_{y=1}^{y=2} \right] dx$$

$$= \int_{x=0}^{x=3} \left[x^2 \left(\frac{2^2}{2} - \frac{1^2}{2} \right) \right] dx$$

$$= \int_{x=0}^{x=3} \frac{3}{2} x^2 \, dx$$ ← now a Calc 1 integral

$$= \frac{3}{2} \frac{x^3}{3} \Big|_{x=0}^{x=3}$$

$$= \frac{1}{2} x^3 \Big|_{x=0}^{x=3}$$

$$= \frac{1}{2} \left(3^3 - 0^3 \right)$$

$$= \boxed{\frac{27}{2}}$$

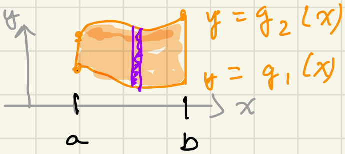
(*) Do "build a porch deck" explanation. Fubini & wood planks. 15.4

Thm. Fubini's Theorem (stronger form)

Let $z = f(x,y)$ with $f: R \to \mathbb{R}$ be continuous.

1. If $R = \{(x,y) \in \mathbb{R}^2 : a \le x \le b \text{ and } g_1(x) \le y \le g_2(x)\}$

TL dydx so 1st y varies x fixed



vertical cross-sections

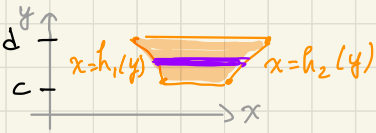
TL: vertically lie wood planks

with $y = g_1(x)$ and $y = g_2(x)$ continuous on $[a,b]$, then

$$\iint_R f(x,y)\, dA = \int_{x=a}^{x=b} \left[\int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\, dy \right] dx$$

2. If $R = \{(x,y) \in \mathbb{R}^2 : c \le y \le d \text{ and } h_1(y) \le x \le h_2(y)\}$

TL dxdy so 1st x varies y fixed



horizontal cross-sections

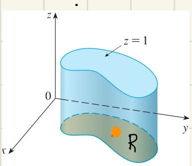
TL: horizontally lie wood

with $x = h_1(y)$ and $x = h_2(y)$ continuous on $[c,d]$ then

$$\iint_R f(x,y)\, dA = \int_{y=c}^{y=d} \left[\int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\, dx \right] dy$$

(*) Important Case In Fubini Thm, if $f(x,y) = 1$ for all $(x,y) \in R$,

then



$\iint_R 1\, dA$ = signed volume of the box w/ base R and height 1

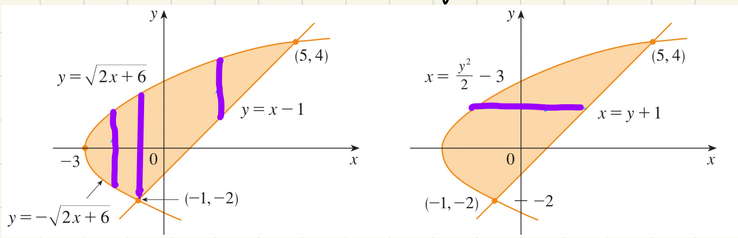
= (height of box) (area of base) = (1) (area of R).

So $\boxed{\iint_R 1\, dA = \text{total (unsigned/actual) area of } R}$

Ex 4 Let R be the region (in xy-plane) bounded by the line $y = x-1$ and the parabola $y^2 = 2x+6$.
Express the (total) area A of R using double integrals in two different ways.

You should know how to go... TL...

Soln · Find points of intersection of $y=x-1$ and $y^2=2x+6$. Will get $(-1,-2)$ and $(5,4)$.
· Sketch R.



Way − dy dx (1st let y vary)   WAY − dx dy (1st let x vary)

Way dydx. $$A = \int_{x=-3}^{x=-1}\left[\int_{y=-\sqrt{2x+6}}^{y=\sqrt{2x+6}} 1\,dy\right]dx + \int_{x=-1}^{x=5}\left[\int_{y=x-1}^{y=\sqrt{2x+6}} 1\,dy\right]dx$$

Way dxdy $$A = \int_{y=-2}^{y=4}\left[\int_{x=\frac{y^2}{2}-3}^{x=y+1} 1\,dx\right]dy$$

$\boxed{\text{Ex 5}}$ Let R be as in Ex 4. Express $\iint_R xy\,dA$ as <u>one</u> double integral.

Soln. $$\iint_R xy\,dA = \int_{y=-2}^{y=4}\left[\int_{x=\frac{y^2}{2}-3}^{x=y+1} xy\,dx\right]dy$$

$\boxed{\text{Ex 6}}$ From Ex 5, evaluate $\iint_R xy \, dA$.

Soln.

$$\iint_R xy\,dA = \int_{y=-2}^{y=4} \left[\int_{x=\frac{y^2}{2}-3}^{x=y+1} xy \, dx \right] dy$$

$$= \int_{y=-2}^{y=4} \left[\frac{x^2}{2} y \Big|_{x=\frac{y^2}{2}-3}^{x=y+1} \right] dy$$

$$= \frac{1}{2} \int_{y=-2}^{y=4} \left[x^2 y \Big|_{x=\frac{y^2}{2}-3}^{x=y+1} \right] dy$$

$$= \frac{1}{2} \int_{y=-2}^{y=4} \left[\underbrace{\left((y+1)^2 y - \left(\frac{y^2}{2}-3\right)^2 y \right)}_{\downarrow \text{ algebra - expand out and collect like terms.}} \right] dy$$

$$= \frac{1}{2} \int_{y=-2}^{y=4} \left[-\frac{y^5}{4} + 4y^3 + 2y^2 - 8y \right] dy$$

$$= \frac{1}{2} \left[-\frac{y^6}{24} + y^4 + \frac{2y^3}{3} - 4y^2 \right] \Bigg|_{y=-2}^{y=4}$$

$\downarrow$ arithmetic

$$= \boxed{36}$$

15.7

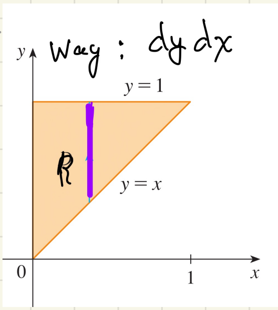
Remark $\int \sin(t^2)\,dt$ is not an "elementary function",
In short, need more than calculus to evaluate $\int \sin t^2\,dt$

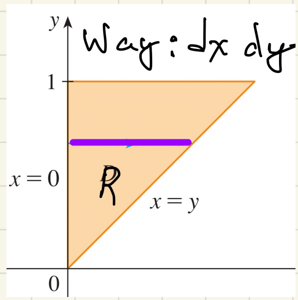
Ex 7 Evaluate $\displaystyle\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx$

Soln

$$\int_0^1\int_x^1 \sin(y^2)\,dy\,dx = \int_{x=0}^{x=1} \left[\int_{y=x}^{y=1} \sin(y^2)\,dy \right] dx$$

TL: we can not evaluate [this part] so let's try switching order of integration





$$= \int_{y=0}^{y=1} \left[\int_{x=0}^{x=y} \sin(y^2)\cdot dx \right] dy$$

$$= \int_{y=0}^{y=1} \left[x\sin(y^2) \Big|_{x=0}^{x=y} \right] dy$$

$$= \int_{y=0}^{y=1} \left[y\sin(y^2) - 0\sin(y)^2 \right] dy$$

$$= \int_0^1 y\sin(y^2)\,dy = -\frac{1}{2}\cos(y^2)\Big|_{y=0}^{y=1}$$

u-du sub. with $u = y^2$ so $du = 2y\,dy$

$$= -\frac{1}{2}\left(\cos 1 - \underbrace{\cos 0}_{1}\right) = \boxed{\frac{1}{2}(1 - \cos 1)}$$