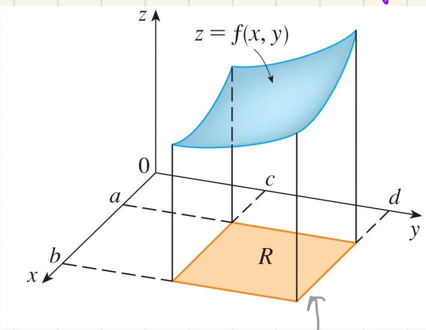
15.1 – 15.3 : Double (Iterated) Integrals over Region R in xy-plane

Goal for Double Integrals . Have: ← TL: Box-ish

a region R in the xy-plane and a function $z = f(x,y)$ with $f: R \to \mathbb{R}^1$.



Want to Express:

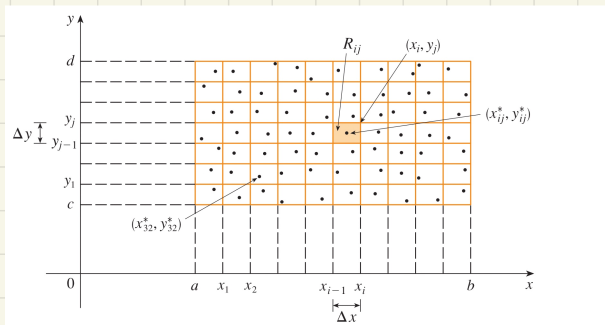
unsigned; actual

1. Total Area of R as: $A = \iint_R 1\, dA$

above R is positive and below R is negative

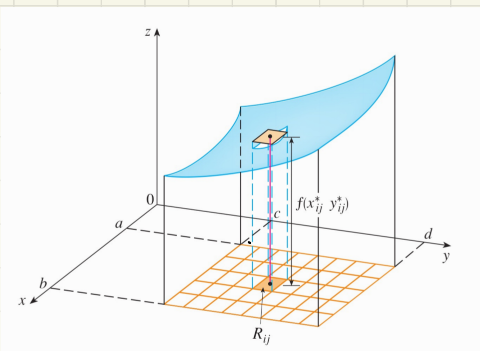
2. Signed Volume of the 3D object bounded by R and the surface $z = f(x,y)$ as: $V = \iint_R f(x,y)\,dA$.

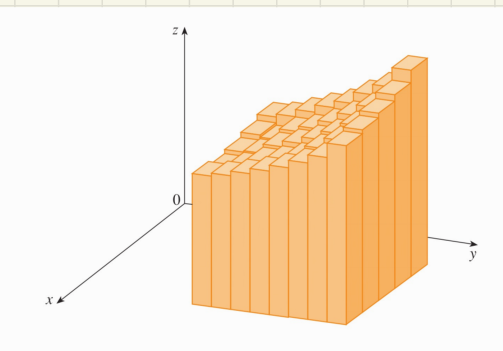
TL: "bottom" & "top" of the 3D object



How to do when $R = [a,b] \times [c,d]$

1. Divide R into "little pieces", eg. subrectangles
2. From each subrectangle $R_{ij} := [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ chose a "sample point" (x_i^*, y_i^*)
3. Over R_{ij}, build a "typical element" (which is a rectanglar box with base R_{ij} and height $f(x_i^*, y_i^*)$)





Volume of typical element (i.e. rect. box) = (height) (area base) = $f(x_i^*, y_j^*)(\Delta x\, \Delta y)$

4. Add together all volumes of typical elements to get:

$$\text{Signed Volume} \approx \sum_j \sum_i f(x_i^*, y_i^*)\, \Delta x\, \Delta y.$$

5. Then take the limit as $\Delta x \to 0$ and $\Delta y \to 0$ to get

$$\text{Signed Volume} = \int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x,y)\, dx\, dy$$

Pictures for $R = [0,2] \times [0,2]$ and $z = 16 - x^2 - y^2$ 15.2

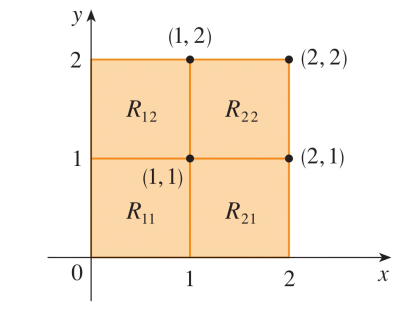


FIGURE 6

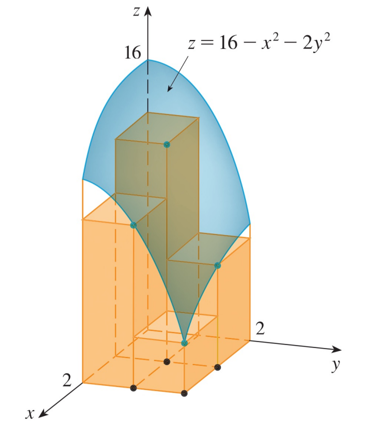


FIGURE 7

note
$f(x,y) = 16 - x^2 - y^2$
is always
above R

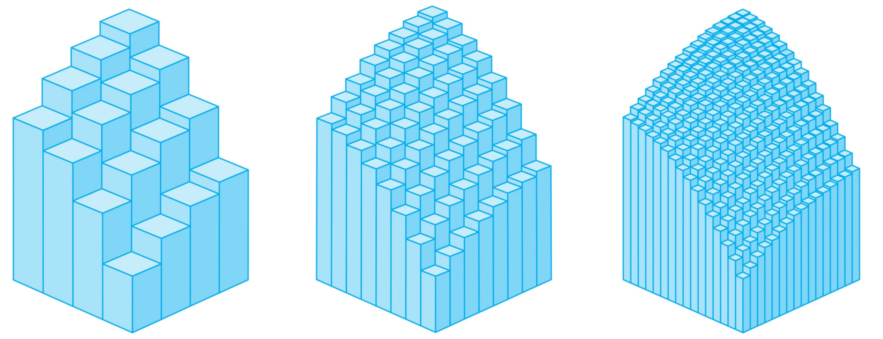


FIGURE 8

The Riemann sum approximations to the volume under $z = 16 - x^2 - 2y^2$ and above R

Source: Calculus by Stewart.

Thm. Fubini – Tonelli for R = rectangle [circa 1908]

Let $z = f(x,y)$ with $f : R \to \mathbb{R}$ be continuous

where $R = [a,b] \times [c,d] \overset{\text{i.e.}}{=} \{(x,y) \in \mathbb{R}^2 \, ; a \le x \le b \text{ and } c \le y \le d\}$.

Then

$$\iint_R f(x,y)\, dA = \text{" signed volume of 3D object bounded by } R \text{ and } z = f(x,y) \text{"}$$

$$= \int_{y=c}^{y=d} \left[\int_{x=a}^{x=b} f(x,y)\, dx \right] dy = \int_{x=a}^{x=b} \left[\int_{y=c}^{y=d} f(x,y)\, dy \right] dx$$

in short

$$= \int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x,y)\, dx\, dy = \int_{x=a}^{x=b} \int_{y=c}^{y=d} f(x,y)\, dy\, dx$$

Remark Fubini says we can "reverse the order of integration"
(i.e. can first $\int$ wrt x then $\int$ wrt y
or can first $\int$ wrt y then $\int$ wrt x)
Sometimes one ordering is much easier to integrate than the other order.

Ex 1. Evaluate $\int_0^3 \int_1^2 x^2 y \, dy\, dx$ better to write $\int_{x=0}^{x=3} \left[\int_{y=1}^{y=2} x^2 y \, dy \right] dx$

Soln $\int_{y=1}^{y=2} x^2 y \, dy = (x^2)\left(\frac{y^2}{2}\right) \Big|_{y=1}^{y=2} = x^2 \left[\frac{2^2}{2} - \frac{1^2}{2} \right] = \frac{3}{2} x^2$.

TL: treat x as a constant

$$\int_{x=0}^{x=3} \left[\int_{y=1}^{y=2} x^2 y \, dy \right] dx = \int_{x=0}^{x=3} \frac{3}{2} x^2 \, dx = \frac{3}{2} \frac{x^3}{3} \Big|_{x=0}^{x=3} \; \textcircled{2}$$

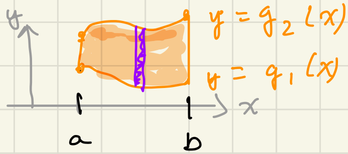
$$\textcircled{2} = \frac{1}{2} x^3 \Big|_{x=0}^{x=3} = \frac{1}{2} \left[3^3 - 0^3 \right] = \boxed{\frac{27}{2}}$$

Thm. Fubini–Tonelli (stronger form)

Let $z = f(x,y)$ with $f: R \to \mathbb{R}$ be continuous.

1. If $R = \{ (x,y) \in \mathbb{R}^2 : a \le x \le b \text{ and } g_1(x) \le y \le g_2(x) \}$ with $y = g_1(x)$ and $y = g_2(x)$ continuous on $[a,b]$

$\boxed{\text{TL } dy\,dx}$



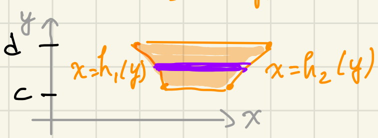
then

$$\iint_R f(x,y)\, dA = \text{"signed volume of 3D object bounded by } R \text{ and } z = f(x,y)\text{"}$$

$$= \int_{x=a}^{x=b} \left[\int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\, dy \right] dx$$

2. If $R = \{ (x,y) \in \mathbb{R}^2 : c \le y \le d \text{ and } h_1(y) \le x \le h_2(y) \}$ with $x = h_1(y)$ and $x = h_2(y)$ continuous on $[c,d]$

$\boxed{\text{TL } dx\,dy}$



then

$$\iint_R f(x,y)\, dA = \text{" signed volume of 3D object bounded by } R \text{ and } z = f(x,y)\text{"}$$

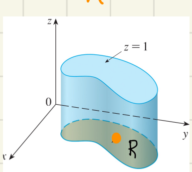
$$= \int_{y=c}^{y=d} \left[\int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\, dx \right] dy$$

(*) Do "build a porch deck" explantion."

Important Case In Fubini Thm, if $f(x,y) = 1$ for all $(x,y) \in R$, then

$$\iint_R 1\, dA = \text{total (unsigned/actual) area of } R.$$

Why?



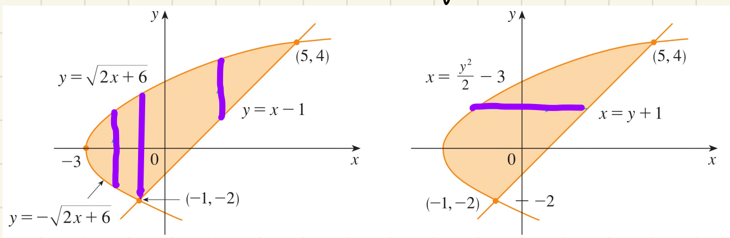
$\iint_R 1\, dA$ = signed volume of the box w/ height 1 and base R

= (height)(base) = 1 (area of R).

Ex 2. Let R be the region (in xy-plane) bounded by the line $y = x-1$ and the parabola $y^2 = 2x+6$. Express the (total) area A of R using double integrals in two different ways.

Soln. • Find points of intersection of $y=x-1$ and $y^2=2x+6$. Will get $(-1,-2)$ and $(5,4)$.

• Sketch R.



Way 1 – $\underline{dy}\,dx$

1st let y vary

WAY 2 – $\underline{dx}\,dy$

1st let x vary

Way 1. $$A = \int_{x=-3}^{x=-1}\left[\int_{y=-\sqrt{2x+6}}^{y=\sqrt{2x+6}} 1\,dy\right]dx + \int_{x=-1}^{x=5}\left[\int_{y=x-1}^{y=\sqrt{2x+6}} 1\,dy\right]dx$$

Way 2. $$A = \int_{y=-2}^{y=4}\left[\int_{x=\frac{y^2}{2}-3}^{x=y+1} 1\,dx\right]dy$$

Ex 3. Let R be as in Ex 2. Express $\iint_R xy\,dA$ as <u>one</u> double integral.

Soln. To get ONE double integral, from Ex. 2, must use Way 2.

$$\iint_R xy\,dA = \int_{y=-2}^{y=4}\left[\int_{x=\frac{y^2}{2}-3}^{x=y+1} xy\,dx\right]dy$$

$\boxed{\text{Ex 4}}$ From Ex 3, evaluate $\iint_R xy\, dA$.

Soln.

$$\iint_R xy\, dA = \int_{y=-2}^{y=4} \left[\int_{x=\frac{y^2}{2}-3}^{x=y+1} xy\, dx \right] dy$$

$$= \int_{y=-2}^{y=4} \left[\frac{x^2}{2} y \Bigg|_{x=\frac{y^2}{2}-3}^{x=y+1} \right] dy$$

$$= \frac{1}{2} \int_{y=-2}^{y=4} \left[x^2 y \Bigg|_{x=\frac{y^2}{2}-3}^{x=y+1} \right] dy$$

$$= \frac{1}{2} \int_{y=-2}^{y=4} \left[\underbrace{\left((y+1)^2 y - \left(\frac{y^2}{2}-3\right)^2 y \right)}_{\downarrow \text{ algebra}} \right] dy$$

$$= \frac{1}{2} \int_{y=-2}^{y=4} \left[-\frac{y^5}{4} + 4y^3 + 2y^2 - 8y \right] dy$$

$$= \frac{1}{2} \left[-\frac{y^6}{24} + y^4 + \frac{2y^3}{3} - 4y^2 \right] \Bigg|_{y=-2}^{y=4}$$

$\downarrow$ arithmetic

$$= \boxed{36}$$

15.7

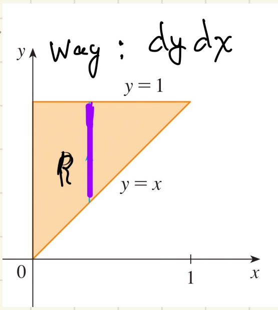
Remark $\int \sin(t^2)\,dt$ is not an "elementary function",
In short, need more than calculus to evaluate $\int \sin t^2 \, dt$

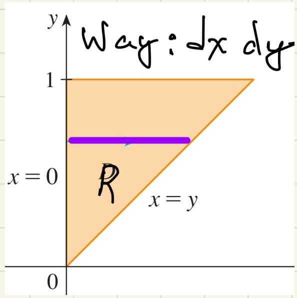
$\boxed{\text{Ex 5}}$ Evaluate $\displaystyle\int_0^1\int_x^1 \sin(y^2)\,dy\,dx$.

Soln

$$\int_0^1\int_x^1 \sin(y^2)\,dy\,dx = \int_{x=0}^{x=1}\underbrace{\left[\int_{y=x^2}^{y=1} \sin(y^2)\,dy\right]}\,dx$$

TL: we can not evaluate [this part] so let's try switching order of integration





$$= \int_{y=0}^{y=1}\left[\int_{x=0}^{x=y} \sin(y^2)\cdot dx\right] dy$$

$$= \int_{y=0}^{y=1}\left[x\sin(y^2)\Big|_{x=0}^{x=y}\right] dy$$

$$= \int_{y=0}^{y=1}\left[y\sin(y^2) - 0\sin(y)^2\right] dy$$

$$= \int_0^1 y\sin(y^2)\,dy = -\frac{1}{2}\cos(y^2)\Big|_{y=0}^{y=1}$$

$$= -\frac{1}{2}\left(\cos 1 - \underbrace{\cos 0}_{1}\right) = \boxed{\frac{1}{2}\left(1-\cos 1\right)}$$