§ 14.7 Extreme Values (Max/Min) and Saddle Points

abbreviations		
max	for	maximum
min	for	minimum
ext.	for	extremum

- Differences btw. function of 1 or 2 variables in PURPLE

$\boxed{\text{Function of 1 variable}}$ (Review Calc 1)

[see Thomas 15th ed page: 223, 222, 1st ¶ 223, 243]

Given $f: D^1 \to \mathbb{R}$ where f's domain D^1 is an interval in $\mathbb{R}$.

Def A critical point of f is a point c in the interior of f's domain where f' is zero or DNE.

1st Der. Thm If f has a local extremum at an interior pt. c of f's domain then c is a critical point of f.

1st Der. Test Local extremum of f can occur only at a critical pt of f or boundary pt of f's domain.

2nd Der. Test Let c be a critical point of f and the 1st and 2nd derivatives of f be continuous in a neighborhood of c.

(1.1) $f'' < 0$ at c <so f is CCD> $\Rightarrow$ c is a local max. of f

(1.2) $f'' > 0$ at c <so f is CCU> $\Rightarrow$ c is a local min of f

(1.3) $f'' = 0$ at c $\Rightarrow$ test is inconclusive

$\boxed{\text{Function of 2 variables.}}$ (Calc 3)

needed def's next page

Given $f: D^2 \to \mathbb{R}$ and $(a,b) \in D^2 \subseteq \mathbb{R}^2$ (f's domain)

Def A critical point of f is a point c in the interior of f's domain where $\vec{\nabla} f$ is $\vec{0}$ or DNE.

1st Der. Thm If f has a local extremum at an interior pt. (a,b) of f's domain then (a,b) is a critical point of f.

1st Der. Test Local extremium of f can occur only at a critical pt of f or boundary pt of f's domain.

Def The discriminate of f is the function $\boxed{D \overset{def}{=} f_{xx} f_{yy} - (f_{xy})^2}$, which is defined for pts where the 2nd partial derv. are continuous in a neighborhood of the point.

Rmk. $D > 0 \Rightarrow f_{xx} f_{yy} > (f_{xy})^2 \Rightarrow f_{xx} f_{yy} > 0 \Rightarrow f_{xx}$ and f_{yy} are both >0 or both <0.

2nd Der. Test. Let (a,b) be a critical point of f and the 1st and 2nd partial der. of f be continuous in a neighborhood of (a,b).

(2.1) $D > 0$ and $f_{xx} < 0$ at (a,b) $\Rightarrow$ (a,b) is local max. of f

(2.2) $D > 0$ and $f_{xx} > 0$ at (a,b) $\Rightarrow$ (a,b) is local min. of f

(2.3) $D < 0$ at (a,b) $\Rightarrow$ (a,b) is saddle point

(2.4) $D = 0$ at (a,b) $\Rightarrow$ test is inconclusive

Rmk. Compare: (2.1) with (1.1) and (2.2) with (1.2) .. think concavity!

Extreme values (Max/Min) and Saddle Points of function of 2 variabes

Given $f: D^2 \to \mathbb{R}$ with $(a,b) \in D^2 \subseteq \mathbb{R}^2$

abbreviations		
max	for	maximum
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ext.	for	extremum

Defs

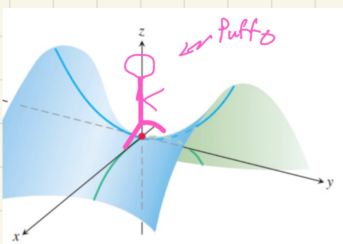
1. A local max occurs at (a,b), and $f(a,b)$ is a local max value, if $f(a,b) \geq f(x,y)$ for each $(x,y) \in D^2 \cap N_r(a,b)$ and for some $r>0$.
2. A local min occurs at (a,b), and $f(a,b)$ is a local min value, if $f(a,b) \leq f(x,y)$ for each $(x,y) \in D^2$.
3. An absolute max occurs at (a,b) and $f(a,b)$ is an absolute max value, if $f(a,b) \geq f(x,y)$ for each $(x,y) \in D^2$
4. An absolute min occurs at (a,b) and $f(a,b)$ is an absolute min value, if $f(a,b) \leq f(x,y)$ for each $(x,y) \in D^2$.

Rmks:
- local is often replacep with relative.
- An extreme point (or extremum) is a max or min point.

5. A saddle point of f occurs at (a,b) if (a,b) is a critical pt. of f but at (a,b) neither a local max nor a local min occurs.

Ex0 Recall (§12.6) the hyperbolic paraboloid (a.k.a. the saddle) $f: \mathbb{R}^2 \to \mathbb{R}$ where

$$f(x,y) = y^2 - x^2 .$$



- $\vec{\nabla} f = \langle -2x, 2y \rangle = \langle 0,0 \rangle \iff (x,y) = (0,0)$

$\Rightarrow (0,0)$ is a critical point of f.

takes some calculations

- $D(0,0) \overset{\text{i.e.}}{=} f_{xx}(0,0)\, f_{yy}(0,0) - (f_{xy}(0,0))^2 = -4 < 0$.
- So 2^{nd} Der.-test $\Rightarrow$ a saddle point occurs at $(0,0)$.

Rmk Memory help. The discriminate of f is

"the Hessian matrix"

$$D \overset{\text{def}}{=} f_{xx} f_{yy} - (f_{xy})^2 \overset{\text{note}}{=} \det \underbrace{\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}}$$

for pts where the 2^{nd} partial derv. are continuous in a neighborhood of the point (so $f_{xy} = f_{yx}$).