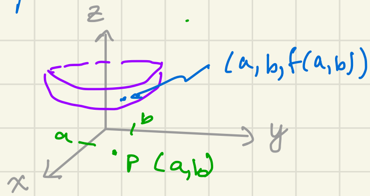
§ 14.7 Extreme Values (Max/Min) and Saddle Points

This Section Setup. $f : D^2 \to \mathbb{R}$ with $(a,b) \in D^2 \subseteq \mathbb{R}^2$



abbreviations

max for maximum
min for minimum.
ext. for extremum

Recall $\vec{\nabla} f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$. In short: $\vec{\nabla} f = \langle f_x, f_y \rangle$.

Def The point $(a,b) \in \mathbb{R}^2$ is a critical point of f provided:
1. (a,b) is in the interior of the domain D^2 of f
2. Either $\vec{\nabla} f(a,b) = \langle 0, 0 \rangle$
   or $\vec{\nabla} f(a,b)$ DNE.

Defs
1. $f(a,b)$ is a local max. of f if $f(a,b) \geq f(x,y)$ for each $(x,y) \in D^2 \cap N_\varepsilon(a,b)$ and for some $\varepsilon > 0$.
2. $f(a,b)$ is a local min. of f if $f(a,b) \leq f(x,y)$ for each $(x,y) \in D^2 \cap N_\varepsilon(a,b)$ and for some $\varepsilon > 0$.
3. $f(a,b)$ is an absolute max. of f if $f(a,b) \geq f(x,y)$ for each $(x,y) \in D$.
4. $f(a,b)$ is an absolute min of f if $f(a,b) \leq f(x,y)$ for each $(x,y) \in D$.
5. (a,b) is a saddle point of a differentiable f at a critical pt. (a,b) if every $N_\varepsilon(a,b)$ contains a pt (x,y) in domain of f with $f(x,y) > f(a,b)$ and also contains a pt $(\tilde{x}, \tilde{y})$ in domain of f w/ $f(\tilde{x}, \tilde{y}) < f(a,b)$.

RmK 1. local is often replaced with relative.
2. An extreme point (or extremum) is a max. or min point.

Thm 1st Derivative Test. If (a,b) is a local extremum of f and $\vec{\nabla} f(a,b)$ exists and (a,b) is an interior point of D, then $\vec{\nabla} f(a,b) = \langle 0, 0 \rangle$.

Recall Setup. $f : D^2 \to \mathbb{R}$ with $(a,b) \in D^2 \subseteq \mathbb{R}^2$

Def If the 2^{nd} order partial derivatives of f exist at (a,b) and the mixed partials are continous in a neighborhood of (a,b) then

- Hessian matrix of f at (a,b) is $H(a,b) = \begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{bmatrix}$

- $D(a,b) = \det\left(H(a,b)\right) = f_{xx}(a,b)\, f_{yy}(a,b) - f_{xy}(a,b)\, f_{yx}(a,b)$.

$= f_{xx}(a,b)\, f_{yy}(a,b) - [f_{xy}(a,b)]^2 \Leftarrow$ Mixed Partial thm

In short

memory trick

$$\boxed{H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \quad \text{and} \quad D = f_{xx} f_{yy} - (f_{xy})^2}$$

Min/Max Tests Summary

1^{st} Der. Test gives:

Extreme values of f can occur only at a:

1. boundary point of domain of f
2. critical point (i.e. an interior pt where $\vec{\nabla} f$ is $\vec{0}$ or DNE)

2^{nd} Der. Test. If (a,b) is a critical point and the 2^{nd} order partial der. are continuous in a neiborhood of (a,b), then

1. $D > 0$ and $f_{xx} < 0$ at (a,b) $\Rightarrow$ (a,b) is local max.
2. $D > 0$ and $f_{xx} > 0$ at (a,b) $\Rightarrow$ (a,b) is local min.
3. $D < 0$ at (a,b) $\Rightarrow$ (a,b) is saddle point
4. $D = 0$ at (a,b) $\Rightarrow$ test is inconclusive

Memory Trick. Compare to 2^{nd} Derv. Test for functions of 1 variable (CalcI)

1. $f''(t) < 0 \Rightarrow$ concave down $\cap$ $\Rightarrow$ max
2. $f''(t) > 0 \Rightarrow$ concave up $\cup$ $\Rightarrow$ min

$\boxed{CP = \text{Critical point}}$

Ex 1. Find absolute extrema for $f(x,y) = 2 + x^2 + y^2$.

BTW: If not given a domain D^2 of f (so $f: D^2 \to \mathbb{R}$), then take the largest domain for which f is defined.
So here, the domain $D^2 = \mathbb{R}^2$.
Review: interior of $\mathbb{R}^2 =$ ______. The boundary of $D^2 =$ ____.

Soln

TL: find Critical Points (CP), i.e. pts in interior of domain w/ $\vec{\nabla} f = \vec{0}$ or $\vec{\nabla} f$ DNE.

- $f_x = 2x$ $\quad f_y = 2y$ $\quad \vec{\nabla} f = \langle 2x, 2y \rangle$

  $\vec{\nabla} f = \langle 0,0 \rangle \iff [2x = 0 \text{ and } 2y = 0] \iff [x = 0 = y]$.
- Note $\vec{\nabla} f$ exists for all $(x,y) \in \mathbb{R}^2 =$ domain of f.
- $\left.\begin{array}{l}\text{Critical points : only } (0,0) \\ \text{boundary points of } \cancel{\text{domain}}\ \mathbb{R}^2 = \emptyset\end{array}\right\} \xRightarrow[\text{Test}]{\text{1st Der.}}$ Extreme value(s) can occur only at $(0,0)$.
- Use 2nd Der. Test w/ $CP = (0,0)$.

  $f_{xx}(x,y) = 2$, $\quad f_{yy}(x,y) = 2$, $\quad f_{xy}(x,y) = 0 \stackrel{\text{note}}{=} f_{yx}(x,y)$
- $D\Big|_{(0,0)} = \left.\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}\right|_{(x,y)=(0,0)} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = (2)(2) - (0)(0) = 4 > 0$
- $f_{xx}(0,0) = 2 > 0$
- So by 2nd Der.v Test $\Rightarrow (0,0)$ local min
- But extreme values only possible at $(0,0)$

  thus $(0,0)$ is where the absolute min. occurs.

  And $f(0,0) = 2 + 0^2 + 0^2 = 2$.
- TL: Let's reread the question to remind us precisely what we are going for.
- The absolute min. value is 2.
  There is not an absolute max. value.
- BTW: We say " the abs. min. value occurs at $(0,0)$.

Next a continuation of Ex 1.

Ex2. Find the max and min values of

$$f(x,y) = 2 + x^2 + y^2$$

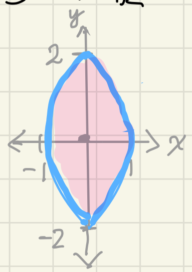
on the set

$$S = \{(x,y) \in \mathbb{R}^2 : x^2 + \frac{y^2}{4} \le 1\}$$

TL $\frac{x^2}{1^2} + \frac{y^2}{2^2} \le 1$

Soln

- Consider $f : S \to \mathbb{R}$, i.e. S = domain of f.
- Sketch S



TL. S is the "pink interior of S" union "blue boundary of S".

$\vec{r}(t) = \langle \cos t, 2\sin t \rangle$, $0 \le t < 2\pi$.

- From Ex 1, the only CP of $f : \mathbb{R}^2 \to \mathbb{R}$ is $(0,0)$. And $(0,0)$ is in interior S.
- Thus the only CP of $f : S \to \mathbb{R}$ is $(0,0)$
- 1^{st} Der. Test $\Rightarrow$ Extrema can occur only at $(0,0)$ or on boundary of S.
- Parameterize the boundary $x^2 + \frac{y^2}{4}$ by : $x(t) = \cos t$, $y(t) = 2\sin t$, $0 \le t < 2\pi$.
- So want to Max/Min the function $z = f(x(t), y(t))$ of 1 variable.
- Consider $g(t) := f(x(t), y(t))$. Solve $dg/dt = 0$ for $0 \le t < 2\pi$

$$\frac{dg}{dt} \overset{CR}{=} \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} = (2x)(-\sin t) + (2y)(2\cos t)$$

$$= (2\cos t)(-\sin t) + 2(2\sin t)(2\cos t) = -2\cos t \sin t + 8 \sin t \cos t$$

$$= 6 \sin t \cos t = 0 \iff t = 0, \pi/2, \pi, 3\pi/2.$$

- Make a chart

	t	(x, y) ($x = \cos t$, $y = 2\sin t$)	$f(x,y)$ ($= 2+x^2+y^2$)	conclusion
critical pt.		$(0,0)$	2	abs. min
boundary pts.	0	$(1,0)$	3	loc. min
	$\pi/2$	$(0,2)$	6	abs. max
	π	$(-1,0)$	3	loc. min
	$3\pi/2$	$(0,-2)$	6	abs. max

Ex 3 Find all local extrema values and saddle points of

$$f(x,y) = x^4 + y^4 - 4xy + 1.$$

Soln • The domain would be (the largest possible... so) $\mathbb{R}^2$.

• So $f: \mathbb{R}^2 \to \mathbb{R}$. The interior of $\mathbb{R}^2 = \mathbb{R}^2$. The bndry of $\mathbb{R}^2 = \emptyset$.

• To find CP (critical points):

$f_x(x,y) = 4x^3 - 4y$, $f_y(x,y) = 4y^3 - 4x$, $\vec{\nabla} f(x,y) = \langle 4x^3 - 4y, 4y^3 - 4x \rangle$

$$\vec{\nabla} f(x,y) = \vec{0} \iff \begin{bmatrix} 4x^3 - 4y = 0 \\ 4y^3 - 4x = 0 \end{bmatrix}$$

$$\iff \begin{bmatrix} x^3 - y = 0 \\ y^3 - x = 0 \end{bmatrix} \Rightarrow y = x^3 \Rightarrow (x^3)^3 - x = 0 \Rightarrow x^9 - x = 0$$

$$0 = x^9 - x = x(x^8 - 1) = x(x^4+1)(x^4-1)$$

$\Rightarrow x = 0$ or $x^4 + 1 = 0$ or $x^4 - 1 = 0$

$x^4 + 1 = 0 \Downarrow x^4 = -1 \Downarrow$ no soln.

$x^4 - 1 = 0 \Downarrow (x^2)^2 = 1 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

Since $y = x^3$, the CP are: $(0,0)$, $(1,1)$, $(-1,-1)$.

• Use 2nd Der. Test on the CP.

$f_{xx}(x,y) = 12x^2$, $f_{yy} = 12y^2$, $f_{xy} = -4$.

$$D(x,y) = \left.\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}\right|_{(x,y)} = \begin{vmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{vmatrix} = 144x^2y^2 - (-4)(-4) = 144x^2y^2 - 16.$$

$D(0,0) = -16 < 0$ so $(0,0)$ is a saddle pt. BTW $f(0,0) = 1$

$D(1,1) = 144 - 16 > 0$ and $f_{xx}(1,1) = 12(1)^2 > 0 \Rightarrow (1,1)$ is a loc. min and $f(1,1) = 1 + 1 - 4 + 1 = -1$.

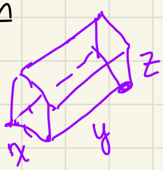
$D(-1,-1) = 144 - 16 > 0$ and $f_{xx}(-1,-1) = 12(-1)^2 > 0 \Rightarrow (1,1)$ is a loc. min and $f(-1,-1) = 1 + 1 - 4(-1)(-1) + 1 = -1$

IL: reread question to recall precisely what is being asked.

• There is one saddle pt. $(0,0)$.
There is one local min. value of -1. (p.s. occurs at $(1,1)$ and $(-1,-1)$)
There is no local max. value.

Ex 4 A rectangular box, without a lid, is to hold 256 cm^3 of sand. Find the dimension of the box that minimizes the surface area of the box (4 sides and bottom, <u>no</u> top).

Soln



$$\text{Surface Area} = \underbrace{2xz}_{\text{sides}} + \underbrace{2yz}_{\text{sides}} + \underbrace{xy}_{\text{bottom}}$$

$$\text{Volume} = xyz = 256 \Rightarrow z = \frac{256}{xy}$$

$$\text{Surface Area} = 2x\left(\frac{256}{xy}\right) + 2y\left(\frac{256}{xy}\right) + xy$$

Want to min $f(x,y) = 512y^{-1} + 512x^{-1} + xy$. Note $x>0$ and $y>0$.

- Find CP.

$f_x(x,y) = -512x^{-2} + y$, $f_y(x,y) = -512y^{-2} + x$.

$$\vec{\nabla} f = \vec{0} \Leftrightarrow \begin{bmatrix} -\frac{512}{x^2} + y = 0 \\ -\frac{512}{y^2} + x = 0 \end{bmatrix} \rightarrow y = \frac{512}{x^2} \rightarrow 0 = \frac{-512}{\left(\frac{512}{x^2}\right)^2} + x = \frac{-x^4}{512} + x = x\left(1 - \frac{x^3}{512}\right)$$

Case $x=0$ cannot physically happen (there would be no box).

So $1 - \frac{x^3}{512} = 0 \Rightarrow x^3 = 512 \Rightarrow x = \sqrt[3]{512} = \sqrt[3]{8^3} = \pm 8$.

Case $x=-8$ cannot physically occur so $x = 8 \Rightarrow y \overset{\text{know}}{=} \frac{512}{x^2} = \frac{8^3}{8^2} = 8$.

- 2^{nd} Der. Test for CP $(8,8)$

$f_{xx} = 2(512)x^{-3}$, $f_{yy} = 2(512)y^{-3}$, $f_{xy} = 1$.

$f_{xx}(8,8) = 2$, $f_{yy}(8,8) = 2$, $f_{xy}(8,8) = 1$

$$D(8,8) = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = (2)(2) - (1)(1) > 0 \text{ and } f_{xx}(8,8) = \frac{2(512)}{8^3} > 0$$

$\Rightarrow$ a min. occurs at $(8,8)$.

TL: Reread the question to recall what precisely was asked.

At $(8,8)$, $z = \frac{256}{xy} = \frac{4 \cdot 8^2}{8 \cdot 8} = 4$.

- $\boxed{\text{For the box bottom, each side should be 8 cm. The box height should be 4 cm.}}$

Recall

$d(\text{pt } Q, \text{ a plane } \mathcal{P} \text{ thru pt. } P \text{ and w/ normal } \vec{n}) = \left| \vec{QP} \cdot \frac{\vec{n}}{\|\vec{n}\|} \right|$

To apply above formula, need a pt. P on plane $\mathcal{P}$.
What if we cannot find a pt P on plane $\mathcal{P}$?

Ex 5 Find a function of 2 variable which can to minimized to provide us with the shortest distance from the point $Q = (1, 0, -2)$ to the plane $\mathcal{P}: x + 2y + z = 4$

$\Downarrow$

Soln. • If (x, y, z) is on the plane $\mathcal{P}$ then $z = 4 - x - 2y$

• $d(Q, \text{ a point } (x,y,z) \text{ on the plane } \mathcal{P})$

$$= d((1,0,-2), (x, y, 4-x-2y))$$

$$= \sqrt{(x-1)^2 + y^2 + [(4-x-2y) - -2]^2}$$

$$= \sqrt{(x-1)^2 + y^2 + (6-x-2y)^2}.$$

• So could min.

$$d(x,y) = \sqrt{(x-1)^2 + y^2 + (6-x-2y)^2}.$$

To make calculation easier, min.

$$\boxed{f(x,y) = (x-1)^2 + y^2 + (6-x-2y)^2}.$$

Will get min. of $z = f(x,y)$ occurs at $\left(\frac{11}{6}, \frac{5}{3}\right)$.

So min. distance will be $\sqrt{f\left(\frac{11}{6}, \frac{5}{3}\right)}$.

Rmk To min. distance btw a point Q and a surface S (which is not a plane), then can also use method in Ex 5.