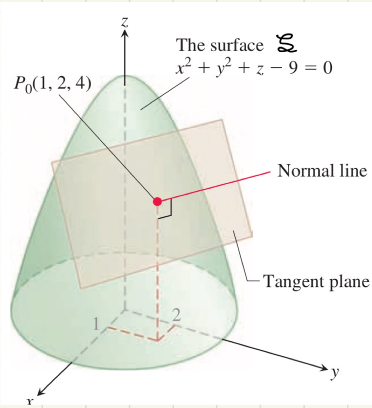
14.6 ngent plane

Defs Normal Line $\mathcal{L}$ and Tangent Plane $\mathcal{P}$ to a level surface $\mathcal{S}$ at P_0

for • a level surface $\mathcal{S}$, i.e. $f(x,y,z) = C$ ← C = a constant
that contains • the point $P_0 = (x_0, y_0, z_0)$ and f is differentiable at P_0
and • $\vec{\nabla} f\,(P_0) \neq \vec{0}$.

Picture for the defs. [Source: Thomas, 15^{th} ed, p 851]



Def Normal line $\mathcal{L}$ contains P_0 and $\vec{\nabla} f|_{P_0}$ is $\parallel$ to (normal line $\mathcal{L}$)

TL: $\vec{\nabla} f|_{P_0}$ gives "the direction of the normal line"

Def Tangent plane $\mathcal{P}$ contains P_0 and $\vec{\nabla} f|_{P_0}$ is normal to (Tangent plane $\mathcal{P}$)
$\perp$

TL: $\vec{\nabla} f|_{P_0}$ gives "a $\vec{n}$ to the tangent plane."

Recall (§ 12.5) →

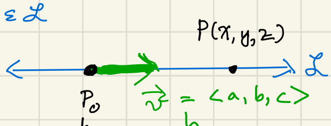
(§ 14.5, p 845)
For Ex 2, recall
$\vec{\nabla} f|_{P_0}$ is
• $\perp$ to a level set of f thru P_0
$\Downarrow$ so
• $\perp$ to any curve $\mathcal{C}$ thru P_0 on a level set of f thru P_0

Line $\mathcal{L}$
thru $P_0 = (x_0, y_0, z_0)$
in direction of $\vec{v} = \langle a,b,c\rangle \neq 0$

Plane $\mathcal{P}$
thru $P_0 = (x_0, y_0, z_0)$
with normal $\vec{n} = \langle a,b,c\rangle \neq \vec{0}$

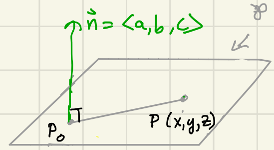
where $P = (x,y,z)$ is an arbitrary point on

$P \in \mathcal{L}$



$x(t) = x_0 + a\,t$
$y(t) = y_0 + b\,t$
$z(t) = z_0 + c\,t$,

$P \in \mathcal{P}$



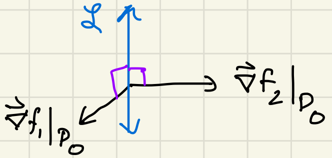
$\vec{n} \cdot \overrightarrow{P_0P} = 0$
$\langle a,b,c\rangle \cdot \langle x - x_0,\ y - y_0,\ z - z_0\rangle = 0$
$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

▶ Now go to next page and come back to Ex 2 after we do Ex. 1.

Ex2. Let the intersection of surface $\mathcal{S}_1$: $f_1(x,y,z) = k_1$ and $\mathcal{S}_2$: $f_2(x,y,z) = k_2$ be a curve $\mathcal{C}$. Fix a point P_0 on $\mathcal{C}$. Then the tangent line $\mathcal{L}$ to $\mathcal{C}$ at P_0 is parallel to ________. Recall class Ex of intersecting 2 planes and get a line. Also Example of (cylinder II) ∩ (plane)

Soln. Know gradient is normal to level sets (§ 14.5, p 845). So

$$\left.\begin{array}{l} \vec{\nabla} f_1|_{P_0} \perp \mathcal{L} \\ \vec{\nabla} f_2|_{P_0} \perp \mathcal{L} \end{array}\right] \Rightarrow$$



$$\Rightarrow \mathcal{L} \parallel \left(\vec{\nabla} f_1|_{P_0} \times \vec{\nabla} f_2|_{P_0}\right)$$

This finishes § 14.6

Setup

Given level surface $S: f(x,y,z) = C$, pt $P_0 = (x_0, y_0, z_0) \in S$, and $\vec{\nabla} f\big|_{P_0} \neq \vec{0}$.

Key: $\vec{\nabla} f\big|_{P_0}$ gives $\left\{ \begin{array}{l} \text{the direction of the normal line} \\ \text{a normal } \vec{n} \text{ to the tangent plane} \end{array} \right\}$ of S at P_0.

- Normal Line $\mathcal{L}$ to $f(x,y,z) = C$ at P_0 is:

$$\boxed{\begin{aligned} x(t) &= x_0 + t\left(f_x\big|_{P_0}\right) \\ y(t) &= y_0 + t\left(f_y\big|_{P_0}\right) \\ z(t) &= z_0 + t\left(f_z\big|_{P_0}\right) \end{aligned}}$$

- Tangent Plane $\mathcal{P}$ to $f(x,y,z) = C$ at P_0 is: $\langle TL: \vec{n} \cdot \vec{PP_0} = 0 \rangle$

$$\boxed{f_x\big|_{P_0}(x - x_0) + f_y\big|_{P_0}(y - y_0) + f_z\big|_{P_0}(z - z_0) = 0}$$

- Tangent Plane $\mathcal{P}$ to $z = f(x,y)$ at $(x_0, y_0, f(x_0, y_0))$ is:

TL: $z = f(x,y) \iff \underbrace{f(x,y) - z}_{} = 0$. And $\vec{\nabla}(f(x,y) - z) = \langle f_x, f_y, -1 \rangle$.

So use above Tangent Plane to eq $(x,y,z) = 0$ to get

$$\boxed{f_x\big|_{(x_0,y_0)}(x - x_0) + f_y\big|_{(x_0,y_0)}(y - y_0) - 1\,(z - f(x_0,y_0)) = 0}$$

Ex 1. Consider the ellipsoid $\dfrac{x^2}{4} + y^2 + \dfrac{z^2}{9} = 3$

and the point $P_0 = (-2, 1, -3)$ on the ellipsoid.

1.1. Find an equation of the normal line $\mathcal{L}$ at P_0 to the ellipsoid.

1.2. Find an equation of the tangent plane $\mathcal{P}$ at P_0 to the ellipsoid.

Soln The ellipsoid is a level surface (of value 3) to $f(x,y,z) = \dfrac{x^2}{4} + y^2 + \dfrac{z^2}{9}$.

$\vec{\nabla} f(x,y,z)\big|_{P_0} = \left\langle \frac{x}{2}, 2y, \frac{2}{9}z \right\rangle\Big|_{(-2,1,-3)} = \langle -1, 2, -2/3 \rangle$

1.1. Normal line $\mathcal{L}$. Use $\vec{\nabla} f\big|_{P_0} \parallel \mathcal{L}$, i.e. $\vec{\nabla} f\big|_{P_0}$ gives direction of $\mathcal{L}$.

So an eq. of $\mathcal{L}$ is

$$\boxed{\begin{aligned} x(t) &= -2 + -1\,t \\ y(t) &= 1 + 2\,t \\ z(t) &= -3 + -\tfrac{2}{3}\,t \end{aligned} \quad, \quad -\infty < t < \infty.}$$

1.2 Tangent Plane $\mathcal{P}$. Use $\vec{\nabla} f\big|_{P_0} \perp \mathcal{P}$, i.e. $\vec{\nabla} f\big|_{P_0}$ is a normal $\vec{n}$ to $\mathcal{P}$.

So an eq of $\mathcal{P}$ is $\boxed{-1(x+2) + 2(y-1) - 2/3\,(z+3) = 0}$

▶ Now do Ex 2 on bottom of page 1.