14.6 Tangent plane

Defs Given:

- a level surface S, i.e. $f(x,y,z) = C$ ← and f is differentiable at P_0

that contains

- the point $P_0 = (x_0, y_0, z_0)$

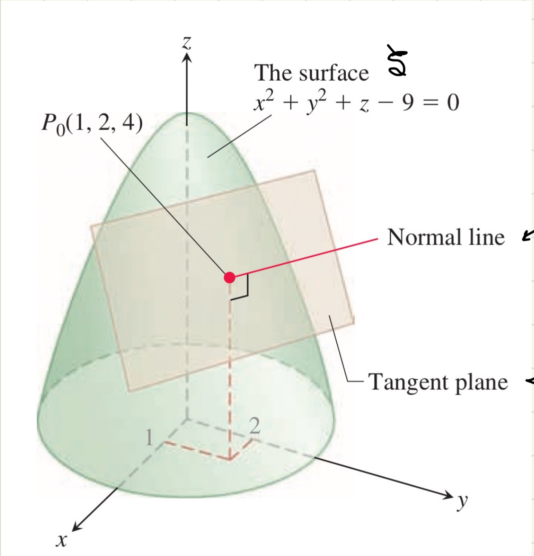
and

- $\vec{\nabla} f(P_0) \neq 0$.

1. The tangent plane to the surface S at point P_0 is the plane thru P_0 that is normal to $\vec{\nabla} f \big|_{P_0}$.

2. The normal line to the surface S at point P_0 is the line thru P_0 that is parallel to $\vec{\nabla} f \big|_{P_0}$.

Picture of a tangent plane and normal line to a surface S at P_0.



Normal line ← thru point P_0 in direction of $\vec{\nabla} f \big|_{P_0}$

Tangent plane ← thru point P_0 with normal vector $\vec{\nabla} f \big|_{P_0}$

Source: Thomas, 15th ed, p 851.

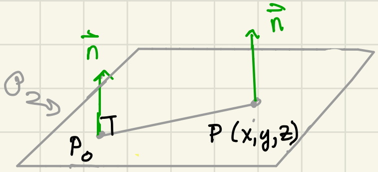
14.6.2

Recall

Plane $\mathcal{P}$	Line $\mathcal{L}$
thru $P_0 = (x_0, y_0, z_0)$ with normal $\vec{n} = \langle a,b,c \rangle$	thru $P_0 = (x_0, y_0, z_0)$ in direction of $\vec{v} = \langle a,b,c \rangle$

where $P = (x,y,z)$ is an arbitrary point on

$P \in \mathcal{P}$

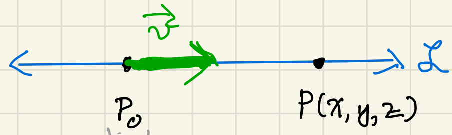


$$\vec{n} \cdot \overrightarrow{PP_0} = 0$$

$$\langle a,b,c \rangle \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0$$

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$P \in \mathcal{L}$



start at P_0, then walk in direction of $\vec{v}$ for time t

$$x(t) = x_0 + a\,t$$
$$y(t) = y_0 + b\,t$$
$$z(t) = z_0 + c\,t$$

$\boxed{\text{So}}$ with $P_0 = (x_0, y_0, z_0)$

- Normal Line $\mathcal{L}$ to $f(x,y,z) = C$ at P_0 is, for $-\infty < t < \infty$:

  TL: $\mathcal{L} \parallel \vec{\nabla} f(P_0) \Rightarrow \mathcal{L}$ is in direction of $\nabla f(P_0)$ so

$$\boxed{\begin{aligned} x(t) &= x_0 + (f_x(P_0))\,t \\ y(t) &= y_0 + (f_y(P_0))\,t \\ z(t) &= z_0 + (f_z(P_0))\,t \end{aligned}}$$

- Tangent Plane $\mathcal{P}$ to $f(x,y,z) = C$ at P_0 is:

  TL: $\mathcal{P} \perp \vec{\nabla} f(P_0) \Rightarrow (\vec{\nabla} f(P_0)) \cdot \overrightarrow{PP} = 0$

$$\boxed{f_x\Big|_{P_0}(x-x_0) + f_y\Big|_{P_0}(y-y_0) + f_z\Big|_{P_0}(z-z_0) = 0}$$

- Tangent Plane $\mathcal{P}$ to $z = f(x,y)$ at $(x_0, y_0, \overbrace{f(x_0,y_0)}^{=z_0})$ is:

  TL: $z = f(x,y) \iff \underbrace{f(x,y) - z = 0}_{g(x,y,z)}$. And $\vec{\nabla}(f(x,y) - z) = \langle f_x, f_y, -1 \rangle$.

  So use above Tangent Plane to $g(x,y,z) = 0$ to get

$$\boxed{f_x\Big|_{(x_0,y_0)}(x-x_0) + f_y\Big|_{(x_0,y_0)}(y-y_0) - 1\,(z-z_0) = 0.}$$

Ex 1. Consider the ellipsoid (a surface $\mathcal{S}$) 14.6.3

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$

and the point $P_0 = (-2, 1, -3)$ on the ellipsoid.

1.1 Find the equation of the tangent plane $\mathcal{P}$ at P_0 to the ellipsoid.

1.2 Find the equation of the normal line $\mathcal{L}$ at P_0 to the ellipsoid.

Soln. The ellipsoid is a level surface (of value 3) to

$$f(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$$

$$\vec{\nabla} f(x, y, z) = \left\langle \frac{x}{2},\ 2y,\ \frac{2}{9} z \right\rangle$$

$$\vec{\nabla} f(-2, 1, -3) = \langle -1,\ 2,\ -2/3 \rangle.$$ ← Key. $\vec{\nabla} f|_{P_0}$: is $\perp \mathcal{P}$ and $\parallel \mathcal{L}$.

1.1. For tangent plane $\mathcal{P}$, use $\vec{\nabla} f|_{P_0} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$.

$$\mathcal{P} : \boxed{-(x+2) + 2(y-1) - 2/3\,(z+3) = 0}$$

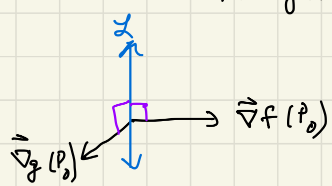
1.2 For normal line $\mathcal{L}$, use $P_0 \in \mathcal{L}$ and $\mathcal{L}$ in direction of $\vec{\nabla} f|_{P_0}$

$$\boxed{\begin{aligned} x(t) &= -2 + -1\,t \\ y(t) &= 1 + 2\,t \\ z(t) &= -3 + -\tfrac{2}{3}\,t \end{aligned} \quad , \quad -\infty < t < \infty}$$

Ex 2. Intersect surfaces $\mathcal{S}_1 : f(x,y,z) = k_1$ and $\mathcal{S}_2 : g(x,y,z) = k_2$.
Get a curve $\mathcal{C}$. Fix a point P_0 on $\mathcal{C}$.
Consider the tangent line $\mathcal{L}$ at P to $\mathcal{C}$. Then $\mathcal{L} \parallel$ ______.

Soln. Recall class Ex (with pipecleaners) of intersecting 2 planes (they are surfaces) to get a line (which is a curve).

Know
$\mathcal{L} \perp \vec{\nabla} f(P_0)$
$\mathcal{L} \perp \vec{\nabla} g(P_0)$
$\Rightarrow$



$$\Rightarrow \mathcal{L} \parallel \left(\vec{\nabla} f|_{P_0} \times \vec{\nabla} g|_{P_0} \right)$$