14.5 Directional Derivative and Gradient Vector

Def 1 If $f: D^2 \to \mathbb{R}$ with $(x_0, y_0) \in D^2 \subseteq \mathbb{R}^2$ and D^2 is open, then

- the gradient vector (for short, gradient) of f is

$$\vec{\nabla} f := \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \quad \text{or} \quad \nabla f$$

common notation to remind us gradient is a vector

book notation

- $$\vec{\nabla} f(x_0, y_0) \overset{\text{i.e.}}{=} \vec{\nabla} f \Big|_{(x_0,y_0)} := \left\langle \frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0) \right\rangle$$

Rmk So $\vec{\nabla} f : D^2 \to \mathbb{V}^2$.

Ex 1 If $f(x,y,z) = x^2 + y^3 + z^4$, then $\vec{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle 2x, 3y^2, 4z^3 \rangle$

Rmk 2 Because ∇ differentiates coordinate-wise, we have:

Algebra Rules for Gradients

1. *Sum Rule:* $\nabla(f+g) = \nabla f + \nabla g$
2. *Difference Rule:* $\nabla(f-g) = \nabla f - \nabla g$
3. *Constant Multiple Rule:* $\nabla(kf) = k\nabla f$ (any number k)
4. *Product Rule:* $\nabla(fg) = \underbrace{f}_{\text{scalar}}\overbrace{\nabla g}^{\text{vector}} + g\nabla f$
5. *Quotient Rule:* $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

Scalar multipliers on left of gradients

Ex 2 $\vec{\nabla}(x^2 y \sin x) = \vec{\nabla}(\overset{f}{(x^2 y)}\overset{g}{(\sin x)})$ ← Product Rule #4

$$= (x^2y)(\vec{\nabla} \sin x) + (\sin x)(\vec{\nabla} x^2 y)$$
$$= x^2 y \langle \cos x, 0 \rangle + \sin x \langle 2xy, x^2 \rangle$$
$$= \langle x^2 y \cos x + 2xy \sin x, x^2 \sin x \rangle.$$

Def 3 Directional Derivative (2 ways)

→ MML/book usually leave out the word 'directional'

- Let $f : D^2 \to \mathbb{R}$ with $(x_0, y_0) \in D^2 \subseteq \mathbb{R}^2$ and D^2 is open.
- Take ANY (nonzero) VECTOR $\vec{v}$ in $\mathbb{R}^2$.

The directional derivative of f at (x_0, y_0) in the direction of $\vec{v} = \langle v_1, v_2 \rangle$, denoted $D_{\frac{\vec{v}}{\|\vec{v}\|}} f(x_0, y_0)$, is the scalar:

recall (§ 12.2 p719) the direction of $\vec{v}$ is $\frac{\vec{v}}{\|\vec{v}\|}$ so when calculating remember to normalize $\vec{v}$ and so use $\frac{\vec{v}}{\|\vec{v}\|}$.

Way 1: using dot product

$$D_{\frac{\vec{v}}{\|\vec{v}\|}} f(x_0, y_0) := \vec{\nabla} f \Big|_{(x_0,y_0)} \cdot \frac{\vec{v}}{\|\vec{v}\|}$$

Way 2: using limits

$$D_{\frac{\vec{v}}{\|\vec{v}\|}} f(x_0, y_0) := \lim_{h \to 0} \frac{f\left(x_0 + h\frac{v_1}{\|\vec{v}\|},\, y_0 + h\frac{v_2}{\|\vec{v}\|}\right) - f(x_0, y_0)}{h}.$$

Picture. In xy-plane, start at (x_0, y_0) and walk in direction of $\vec{v} := \langle v_1, v_2 \rangle$

start (x_0, y_0) ↘ $\frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle v_1, v_2 \rangle}{\|\vec{v}\|}$ $\Rightarrow$ so walking along with position vector

$$\vec{r}(t) = \Big\langle \underbrace{x_0 + t\frac{v_1}{\|\vec{v}\|}}_{x(t)},\ \underbrace{y_0 + t\frac{v_2}{\|\vec{v}\|}}_{y(t)} \Big\rangle \Rightarrow \vec{r}\,'(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$$

$$\Rightarrow \vec{r}\,'(t) = \left\langle \frac{v_1}{\|\vec{v}\|}, \frac{v_2}{\|\vec{v}\|} \right\rangle$$

- The limit in **Way 2** says the

directional der. of f at (x_0,y_0) in the direction of $\vec{v}$ is the rate of change at $z = f(x,y)$ at (x_0, y_0) as we move in the xy-plane from (x_0, y_0) in the direction of $\vec{v}$ (so along $\vec{r}$)

- So looking at rate of change of $z = f(x(t), y(t))$ ⟨a function of t⟩ where $x(t) = x_0 + t\frac{v_1}{\|\vec{v}\|}$ and $y(t) = y_0 + t\frac{v_2}{\|\vec{v}\|}$. So looking at $\frac{df}{dt}$ at (x_0, y_0). Well

$$\frac{df}{dt} \overset{CR}{=} \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{\partial f}{\partial x}\frac{v_1}{\|\vec{v}\|} + \frac{\partial f}{\partial y}\frac{v_2}{\|\vec{v}\|} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \left\langle \frac{v_1}{\|\vec{v}\|}, \frac{v_2}{\|\vec{v}\|} \right\rangle = \vec{\nabla} f \cdot \frac{\vec{v}}{\|\vec{v}\|}$$

so $\frac{df}{dt}(x_0, y_0) = \vec{\nabla} f \Big|_{(x_0, y_0)} \cdot \frac{\vec{v}}{\|\vec{v}\|}$ ← **Way 1** for $D_{\frac{\vec{v}}{\|\vec{v}\|}} f(x_0, y_0)$

☺ Way 1 = Way 2, by the Chain Rule!

Ex 3 Find the directional derivative of $f(x,y) = 9 - x^2 - y^2$ at the point $(1,1)$ in the direction of $\vec{v} = \langle -1, -6 \rangle$.

Soln

$$D_{\frac{\vec{v}}{\|\vec{v}\|}} f(1,1) = \vec{\nabla} f \Big|_{(1,1)} \cdot \frac{\vec{v}}{\|\vec{v}\|} = \langle -2x, -2y \rangle \Big|_{(1,1)} \cdot \frac{\langle -1, -6 \rangle}{\sqrt{37}}$$

$$= -2 \langle 1, 1 \rangle \cdot \frac{-1}{\sqrt{37}} \langle 1, 6 \rangle = \frac{2}{\sqrt{37}} (1+6) = \boxed{\frac{14}{\sqrt{37}}}$$

Ex 4 Desmos 14.5.1 (goes with Ex 3).

Question 1: In Ex 3, why is the $D_{\frac{\vec{v}}{\|\vec{v}\|}} f(1,1) > 0$?

Rmk Start with a function $z = f(x,y)$ and a point $P = (x_0, y_0)$ w/ $\vec{\nabla} f(x_0, y_0) \neq \vec{0}$. Think of letting a UNIT vector $\vec{u}$ vary. Then

$$D_{\vec{u}} f(x_0, y_0) = \vec{\nabla} f \Big|_P \cdot \vec{u} = \| \vec{\nabla} f(x_0, y_0) \| \cos (\measuredangle \text{ btw } \vec{\nabla} f|_P \text{ and } \vec{u}).$$

So

$D_{\vec{u}} f(x_0, y_0)$
- is the largest when $\measuredangle$ btw $\vec{\nabla} f(x_0,y_0)$ and $\vec{u}$ = 0.
- is 0 when $\measuredangle$ btw $\vec{\nabla} f(x_0,y_0)$ and $\vec{u}$ = $\frac{\pi}{2}$.
- is the smallest when $\measuredangle$ btw $\vec{\nabla} f(x_0,y_0)$ and $\vec{u}$ = π.

Thus

- f increases most rapidly in the direction of $\vec{\nabla} f|_P$, w/ rate of change $= \| \vec{\nabla} f|_P \|$.
- f does not vary if $\vec{\nabla} f|_P \perp \vec{u}$.
- f decreases most rapidly in the direction of $-\vec{\nabla} f|_P$, w/ rate of change $= -\| \vec{\nabla} f|_P \|$.

Ex 5 In Ex 4, at $P = (1,1)$ the function $z = f(x,y)$ increases most rapidly

(a) in the direction $\vec{u}$ = __________. ← need unit vector!

Soln (a) $\vec{\nabla} f|_P = \langle -2x, -2y \rangle \Big|_{(1,1)} = \langle -2, -2 \rangle$.

$\| \vec{\nabla} f|_P \| = \| -2 \langle 1,1 \rangle \| = 2 \| \langle 1,1 \rangle \| = 2\sqrt{2}$.

$$\vec{u} = \frac{\langle -2, -2 \rangle}{2\sqrt{2}} = \langle -\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}} \rangle = \boxed{\langle -\tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2} \rangle}$$

and (b) increases at rate of __________

Soln (b) rate of change is $D_{\vec{u}} f(1,1) = \vec{\nabla} f|_{(1,1)} \cdot \vec{u} = \langle -2, 2 \rangle \cdot \langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \rangle = \boxed{2\sqrt{2}}$

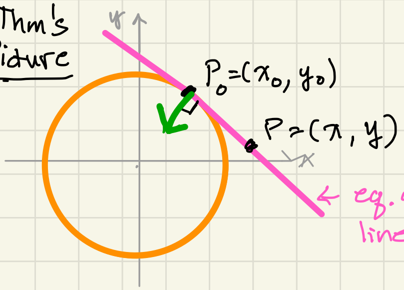
↑ need UNIT vector

Ex 6 Desmos 14.5.2. Goes with Ex 5.

Thm 4 (p 845 <MT 167>)

$$\boxed{\text{If } z=f(x,y) \text{ has a continuous nonzero } \vec{\nabla} f \text{ at } P_0=(x_0,y_0), \text{ then } \vec{\nabla} f\big|_{P_0} \text{ is normal to (the level curve thru } P_0). \text{ i.e. } \perp \text{ to the tangent line to level curve thru } P_0)}$$

Thm's Picture



Have $\vec{\nabla} f\big|_{P_0} \perp \overrightarrow{P_0P}$

$\Rightarrow \left\langle f_x\big|_{P_0}, f_y\big|_{P_0} \right\rangle \cdot \langle x-x_0, y-y_0\rangle = 0$

← eq. of this line is ⟹ $\boxed{f_x\big|_{P_0}(x-x_0) + f_y\big|_{P_0}(y-y_0) = 0}$

Ex 7 In Ex 6 ... Desmos 14.5.2 ...

Find an equation of the tangent line to the level set of $f(x,y)=9-x^2-y^2$ at $P=(1,1)$

May use, from Ex 5, that $\vec{\nabla} f\big|_{(1,1)} = \langle -2,-2\rangle$.

Soln $0 = \vec{\nabla} f\big|_{(1,1)} \cdot \langle x-1, y-1\rangle = \langle -2,-2\rangle \cdot \langle x-1, y-1\rangle = -2(x-1) + {}^-2(y-1)$.

$\Rightarrow -2(x-1) + {}^-2(y-1) = 0 \xRightarrow{\text{Alg}} (x-1)+(y-1)=0 \Rightarrow \boxed{x+y=2}$

Rmk. Thm 4 follows from the chain rule:

a constant $c = f(x(t), y(t))$. Next d/dt both sides to get

$$0 = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle = \vec{\nabla} f \cdot \vec{r}\,'(t)$$

So $\vec{\nabla} f$ is normal to tang. vector $\vec{r}\,'$ so is normal to the level curve,

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Can now do 14.5 HW

Thm 5 (p 849)

$$\boxed{\begin{array}{l} \text{If } w=f(x,y,z) \text{ is a diff. scalar-valued function} \\ \text{and } \vec{r}(t) = \langle x(t), y(t), z(t)\rangle \text{ is a smooth path,} \\ \text{then } w = f(x(t), y(t), z(t)) \text{ is differentiable (w.r.t. } t) \\ \text{and } \dfrac{d}{dt} f(t) = \vec{\nabla} f(\vec{r}(t)) \cdot \vec{r}\,'(t). \end{array}}$$

Why? $\dfrac{df}{dt} \underset{\text{Chain Rule}}{=} \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial f}{\partial z}\dfrac{dz}{dt} = \left\langle \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}\right\rangle \cdot \left\langle \dfrac{dx}{dt}, \dfrac{dy}{dt}, \dfrac{dz}{dt}\right\rangle$

This finishes 14.5. Any questions?