14.5 Directional Derivative and Gradient Vector

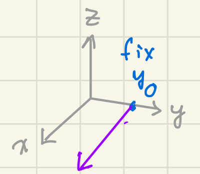
(left out, but used, in Th 15. In Th 13 p 817 § 14.3)

Def. A trace curve (for short, trace) in $\mathbb{R}^3$ is a curve in $\mathbb{R}^3$ obtained by intersecting:
a surface in $\mathbb{R}^3$ (e.g. the graph of $z = f(x,y)$) with
a plane in $\mathbb{R}^3$

Directional Derivative

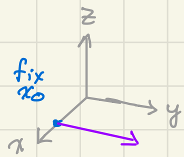
- Let $f : D^2 \to \mathbb{R}^2$ with $(x_0, y_0) := P_0 \in D^2 \subseteq \mathbb{R}^2$ and $N_\varepsilon((x_0, y_0)) \subseteq D^2$ for some $\varepsilon > 0$.

- Recall Partial Derivatives



- $$\frac{\partial f}{\partial x}(x_0, y_0) := \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

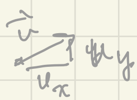
  is the rate of change of f at (x_0, y_0) in the direction of $\vec{i}$



- $$\frac{\partial f}{\partial y}(x_0, y_0) := \lim_{h \to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h}$$

  is the rate of change of f at (x_0, y_0) in the direction of $\vec{j}$

- Now take any unit vector $\vec{u} = \langle u_x, u_y \rangle$



Def The directional derivative of f at (x_0, y_0) in the direction of $\vec{u}$ is

$$D_{\vec{u}} f(x_0, y_0) := \lim_{h \to 0} \frac{f(x_0 + hu_x, y_0 + hu_y) - f(x_0, y_0)}{h}.$$

So $D_{\vec{u}} f(x_0, y_0)$ is the rate of change of f at (x_0, y_0) in the direction of $\vec{u}$

Def For any (nonzero) vector $\vec{v}$
The directional derivative of f at (x_0, y_0) in the direction of $\vec{v}$ is

$$\boxed{D_{\frac{\vec{v}}{\|\vec{v}\|}} f(x_0, y_0)}$$

← need to normalize $\vec{v}$!

14,5,2

Geometric Interpretation of Directional Derivative

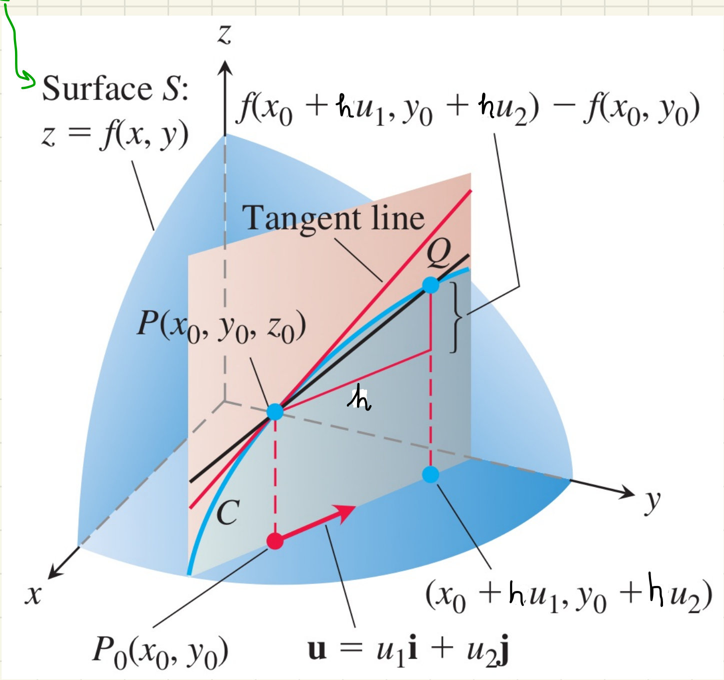
$D_{\vec{u}} f(x_0, y_0)$

when $\|\vec{u}\| = 1$.

for $\underbrace{z = f(x, y)}_{\text{graph is a surface in } \mathbb{R}^3}$ and $\vec{u} = \langle u_1, u_2 \rangle$

graph is a surface in $\mathbb{R}^3$
get surface S

direction in xy-plane



The slope of the trace curve C at P_0 is $\lim_{Q \to P}$ slope (PQ); this is

the directional derivative $(D_{\mathbf{u}} f)_{P_0}$.

- How to compute $D_{\vec{u}} f$? ..... use gradient (next page)

Def If $f: D^2 \to \mathbb{R}$ with $P=(x,y) \in D^2 \subseteq \mathbb{R}^2$, then

- the gradient vector (for short, gradient) of f is

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$ ← book / MML Short hand

$$\vec{\nabla} f \Big|_P = \vec{\nabla} f(x,y) = \left\langle \frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \right\rangle$$ ← Prof. G. long version

remind us $\vec{\nabla}$ is a vector and $\vec{\nabla} f : D^2 \to \mathcal{V}^2$.

Ex. If $f(x,y,z) = x^2+y^3+z^4$, then $\vec{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle 2x, 3y^2, 4z^3 \rangle$.

Recall Directional Derivative is defined by a limit.

Thm Directional Derivative as a Dot Product ← easier than a limit.

- Let $f: D^2 \to \mathbb{R}^2$ with $(x_0, y_0) := P_0 \in D^2 \subseteq \mathbb{R}^2$ and $N_\varepsilon((x_0,y_0)) \subseteq D^2$ w/ $\varepsilon > 0$.
- Take ANY (non zero) VECTOR $\vec{v}$ in $\mathbb{R}^2$.

(book often leave out this word)

remember to normalize

The directional derivative of f at P_0 in the direction of $\vec{v}$ is

$$\boxed{D_{\frac{\vec{v}}{\|\vec{v}\|}} f \Big|_{P_0} = \vec{\nabla} f \Big|_{P_0} \cdot \frac{\vec{v}}{\|\vec{v}\|}}$$ ← general case

So if $\vec{v}$ is unit vector (i.e. $\|\vec{v}\| = 1$) then

$$\boxed{D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u}}$$ ← unit vector case

Recall $D_{\vec{u}} f(x_0, y_0)$ is the rate of change of f at (x_0, y_0) in the direction of $\vec{u}$

Recall Example from § 14.3 (and Desmos)

Ex 1. Let

$$f(x,y) = \sqrt{9-x^2-y^2}$$

calculus friendly $(9-x^2-y^2)^{1/2}$

1.1. $\frac{\partial f}{\partial x}(x,y) = \frac{-x}{\sqrt{9-x^2-y^2}}$

1.2 $f_y(x,y) = \frac{-y}{\sqrt{9-x^2-y^2}}$

1.3 $f_y(1,2) = -1$

Let continue with this example

1.4 $f_x(1,2) = \frac{-1}{\sqrt{9-1-4}} = \frac{-1}{\sqrt{4}} = -\frac{1}{2}$

1.5 $\vec{\nabla} f(x,y) = \langle f_x, f_y \rangle = \left\langle \frac{-x}{\sqrt{9-x^2-y^2}}, \frac{-y}{\sqrt{9-x^2-y^2}} \right\rangle$

1.6 $\vec{\nabla} f(1,2) = \langle f_x(1,2), f_y(1,2) \rangle = \langle -\frac{1}{2}, -1 \rangle$

1.7 Find the (directional) derivative of f at the point $(1,2)$ in the direction of $\vec{v} = \langle 3,4 \rangle$.

Soln

Want → $\left(D_{\frac{\vec{v}}{\|\vec{v}\|}} f \right) \Big|_{(1,2)}$

Know $\underbrace{\vec{\nabla} f \Big|_{(1,2)}}_{\downarrow \text{Ex 1.5}} \cdot \frac{\vec{v}}{\|\vec{v}\|}$

$= \langle -\frac{1}{2}, -1 \rangle \cdot \frac{\langle 3,4 \rangle}{\sqrt{3^2+4^2}}$

$= \frac{1}{5}\left((-\frac{1}{2})(3) + (-1)(4) \right) = \frac{1}{5}\left(\frac{-11}{2} \right)$

$= \boxed{\frac{-11}{10}}$

Explore: Have/fix a function $z = f(x,y)$ and a point P.

Think a letting vary the : unit direction $\vec{u}$

Get

$$D_{\vec{u}} f \big|_P = \vec{\nabla} f \big|_P \cdot \vec{u} = \|\vec{\nabla} f|_P\| \cos\underbrace{(\measuredangle\ \vec{\nabla} f|_P \text{ and } \vec{u})}$$

$(\measuredangle\ \vec{\nabla} f|_P$ and $\vec{u})$ varies from $0 \xrightarrow{\text{to}} \frac{\pi}{2} \to \pi$

so $\cos(\measuredangle\ \vec{\nabla} f|_P$ and $\vec{u})$ varies from $1 \xrightarrow{\text{to}} 0 \to -1$

so $D_{\vec{u}} f|_P$ varies from $\|\vec{\nabla} f|_P\| \to 0 \to -\|\vec{\nabla} f|_P\|$

max, min

so as the direction $\vec{u}$ varies ~~get~~:

Recall: $D_{\vec{u}} f(x_0, y_0)$ is the rate of change of f at (x_0, y_0) in the direction of $\vec{u}$.

Thus at a point P

- f increases most rapidly in the direction of $\nabla f|_P$, w/ rate of change $= \|\vec{\nabla} f|_P\|$.
- f decreases most rapidly in the direction of $-\nabla f|_P$, w/ rate of change $= -\|\vec{\nabla} f|_P\|$.

  and if $\vec{\nabla} f|_P \neq \vec{0}$.
- f does not vary if $\vec{\nabla} f|_P \perp \vec{u}$.

Revisit Ex 1. $f(x,y)=\sqrt{9-x^2-y^2}$ and $P=(1,2)$ 14.5.6

HAD

1.5 $\vec{\nabla} f(x,y) = \langle f_x, f_y \rangle = \left\langle \dfrac{-x}{\sqrt{9-x^2-y^2}}, \dfrac{-y}{\sqrt{9-x^2-y^2}} \right\rangle$

1.6 $\vec{\nabla} f(1,2) = \langle f_x(1,2), f_y(1,2) \rangle = \langle -\tfrac{1}{2}, -1 \rangle$

New:

1.8 At the point $P=(1,2)$, $z=f(x,y)$ is increasing most rapidly in the direction of $\boxed{\left\langle -\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right\rangle}$ note $-\frac{1}{\sqrt{5}}\langle 1,2 \rangle$

$\vec{v} = \langle -\tfrac{1}{2}, -1 \rangle \Rightarrow \|\vec{v}\| = \sqrt{\tfrac{1}{4}+1} = \frac{\sqrt{5}}{2} \Rightarrow \frac{\vec{v}}{\|\vec{v}\|} = \left\langle -\frac{1}{2}\frac{2}{\sqrt{5}}, -1\left(\frac{2}{\sqrt{5}}\right) \right\rangle$

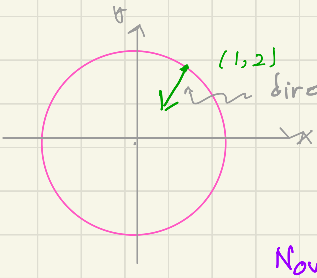
Key observation. $f(1,2) = \sqrt{9-1-4} = 2$

So $(1,2)$ is on the level curve $f(x,y)=2$, i.e. $\sqrt{9-x^2-y^2}=2$

$\Updownarrow$ A

$x^2+y^2=5$

ps $\sqrt{5} \approx 2.2$



Can review at Desmos 14.1.1 on level sets.

Now Look at Desmos 14.5.1 on gradient

Thm p845 For a differentiable $z=f(x,y)$ and a point $P_0=(x_0,y_0)$

If $\vec{\nabla} f|_{P_0} \neq \vec{0}$, then

$\vec{\nabla} f|_{P_0}$ is normal to (the level curve thru P_0).

i.e. $\perp$ to tang. line to level curve

Let's see what can get from this Thm... $\longrightarrow$

So Thm gives:

$$\underbrace{\text{tangent line } \mathcal{L} \text{ at } P_0 \overset{\parallel}{(x_0, y_0)} \text{ to the level set } f(x,y) = f\big|_{P_0}}\ \perp\ \vec{\nabla} f\big|_{P_0}$$

line $\mathcal{L}$ is contained in a plane $\mathcal{P}$ that:

- thru point $(x_0, y_0, 0)$
- has normal vector $\langle f_x\big|_{P_0}, f_y\big|_{P_0}, 0\rangle$.

So the equation of $\mathcal{P}$ (in 3D) is:

$$\left(f_x\big|_{P_0}\right)(x - x_0) + \left(f_y\big|_{P_0}\right)(y - y_0) + 0\,(z - 0) = 0$$

So the equation of the tangent line (in xy-plane) to the level curve $f(x,y) = f(x_0, y_0)$ is:

$$\boxed{\left(f_x(x_0, y_0)\right)(x - x_0) + \left(f_y(x_0, y_0)\right)(y - y_0) = 0.}$$

in short

$$\vec{\nabla} f\big|_{P_0} \cdot \langle x - x_0, y - y_0\rangle = 0.$$

Ex 1.9 Find the equation of the tangent line $\mathcal{L}$ to the level set $f(x,y) = 2$ at the point $P_0 = (1,2)$

Soln $\vec{\nabla} f(1,2) \overset{\text{Ex 1.6}}{=} \langle -\frac{1}{2}, -1\rangle$.

$$\langle -\tfrac{1}{2}, -1\rangle \cdot \langle x - 1, y - 2\rangle = 0$$

$$\boxed{-\tfrac{1}{2}(x - 1) + (y - 2) = 0}$$

Rmk: in slope-intercept form, $\mathcal{L}$ is $f(x) = -\frac{1}{2}x + \frac{5}{2}$.

Recall

14.5.8

Thm p845 For a differentiable $z = f(x,y)$ and a point $P_0 = (x_0, y_0)$

If $\vec{\nabla} f|_{P_0} \neq \vec{0}$, then

$\vec{\nabla} f|_{P_0}$ is normal to (the level curve thru P_0).

Why? Follows from chain rule see page 845.

a constant $c = f(x(t), y(t))$. Next d/dt both sides to get

$$0 = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \vec{\nabla} f \cdot r'(t)$$

So $\vec{\nabla} f$ is normal to tang. vector r' so is normal to the level curve.

New and similar

Thm p849 If $w = f(x,y,z)$ is a diff. scalar-valued function and $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ is a smooth path,

then $w = f(x(t), y(t), z(t))$ is differentiable (w.r.t. t)

and $$\frac{d}{dt} f(t) = \vec{\nabla} f(\vec{r}(t)) \cdot \vec{r}\,'(t).$$

Why? $$\frac{df}{dt} \underset{\text{Chain Rule}}{=} \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$$

Remark: because ∇ differentiates coord-wise:

Algebra Rules for Gradients

1. *Sum Rule:* $\nabla(f+g) = \nabla f + \nabla g$
2. *Difference Rule:* $\nabla(f-g) = \nabla f - \nabla g$
3. *Constant Multiple Rule:* $\nabla(kf) = k\nabla f$ (any number k)
4. *Product Rule:* $\nabla(fg) = \underbrace{f}_{\text{scalar}}\underbrace{\nabla g}_{\text{vector}} + g\nabla f$
5. *Quotient Rule:* $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

Scalar multipliers on left of gradients