

Ex 1. Let $w = x^2y + y + xz$ and

$$x = \cos \theta, \quad y = \sin \theta, \quad z = \theta^2.$$

$$\begin{array}{ccccc} & & w & & \\ & \swarrow & \vdots & \searrow & \\ x & & y & & z \\ \vdots & & \vdots & & \vdots \\ \theta & & \theta & & \theta \end{array}$$

1.1. Express $\frac{dw}{d\theta}$ as a function of θ .

1.2. Evaluate $\frac{dw}{d\theta}$ when $\theta = \frac{\pi}{2}$.

Soln 1.1.

$$\begin{aligned} \frac{dw}{d\theta} &\stackrel{\text{CR}}{=} \frac{\partial w}{\partial x} \frac{dx}{d\theta} + \frac{\partial w}{\partial y} \frac{dy}{d\theta} + \frac{\partial w}{\partial z} \frac{dz}{d\theta} \\ &= (2xy + z)(-\sin \theta) + (x^2 + 1)(\cos \theta) + x(2\theta) \\ &= (2 \cos \theta \sin \theta + \theta^2)(-\sin \theta) + (\cos^2 \theta + 1) \cos \theta + 2\theta \cos \theta \\ &= \boxed{-2 \cos \theta \sin^2 \theta - \theta^2 \sin \theta + \cos^3 \theta + \cos \theta + 2\theta \cos \theta}. \end{aligned}$$

Soln 1.2. Know $\cos(\pi/2) = \dots 0 \dots$ and $\sin(\pi/2) = \dots 1 \dots$. So

$$\left. \frac{dw}{d\theta} \right|_{\theta=\pi/2} = 0 - \left(\frac{\pi}{2}\right)^2 \sin \frac{\pi}{2} + 0 + 0 + 0 = \boxed{-\frac{\pi^2}{4}}.$$

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Ex 2. Let $w = xy + \ln(xy)$, where $x = rt$ and $y = r + t^2$. Find

$$\left. \frac{\partial w}{\partial t} \right|_{(r,t)=(2,1)}.$$

$$\begin{array}{ccccc} & & w & & \\ & \swarrow & & \searrow & \\ x & & & & y \\ \swarrow \quad \searrow & & & & \swarrow \quad \searrow \\ r & t & & r & t \end{array}$$

Soln.

$$\begin{aligned} \frac{\partial w}{\partial t} &\stackrel{\text{CR}}{=} \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} \\ &= \left(y + \frac{y}{xy}\right) r + \left(x + \frac{x}{xy}\right) (2t) \\ &= \left(y + \frac{1}{x}\right) r + \left(x + \frac{1}{y}\right) (2t). \end{aligned}$$

When $(r, t) = (2, 1)$, we have $x = rt = \dots 2 \dots$ and $y = r + t^2 = \dots 3 \dots$. So

$$\begin{aligned} \left. \frac{\partial w}{\partial t} \right|_{(r,t)=(2,1)} &= \left(3 + \frac{1}{2}\right) (2) + \left(2 + \frac{1}{3}\right) (2) \\ &= 6 + 1 + 4 + \frac{2}{3} = 11\frac{2}{3} = \boxed{\frac{35}{3}}. \end{aligned}$$

Implicit Differentiation Examples

Ex 3. Find $\frac{dy}{dx}$ if $x^3 + y^3 = 6xy$.

Soln.

$$x^3 + y^3 = 6xy \iff x^3 + y^3 - 6xy = 0.$$

So let

$$F(x, y) = \dots\dots\dots x^3 + y^3 - 6xy \dots\dots\dots.$$

Then

$$\begin{aligned} \frac{dy}{dx} & \underset{\text{diff.}}{\overset{\text{implicit}}{=}} -\frac{F_x}{F_y} \\ &= -\frac{3x^2 + 0 - 6y}{0 + 3y^2 - 6x} \\ &= -\frac{3x^2 - 6y}{3y^2 - 6x} = \boxed{-\frac{x^2 - 2y}{y^2 - 2x}}. \end{aligned}$$

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Ex 4. Find $\frac{\partial z}{\partial x}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

Soln.

$$x^3 + y^3 + z^3 + 6xyz = 1 \iff x^3 + y^3 + z^3 + 6xyz - 1 = 0.$$

So let

$$F(x, y, z) = \dots\dots\dots x^3 + y^3 + z^3 + 6xyz - 1 \dots\dots\dots.$$

Then

$$\begin{aligned} \frac{\partial z}{\partial x} & \underset{\text{diff.}}{\overset{\text{implicit}}{=}} -\frac{F_x}{F_z} \\ &= -\frac{3x^2 + 0 + 0 + 6yz + 0}{0 + 0 + 3z^2 + 6xy + 0} \\ &= -\frac{3x^2 + 6yz}{3z^2 + 6xy} \\ &= \boxed{-\frac{x^2 + 2yz}{z^2 + 2xy}}. \end{aligned}$$