14-4 Chain Rule (CR)

diff = differentiable
Thm = Theorem (a fact)
Cor = Corollary (follows from a Thm)

Thm 1 If: $w = f(x, y)$ is diff
$x = x(t)$ and $y = y(t)$ are diff w.r.t. t
then $w = f(x(t), y(t))$ is diff. w.r.t t

and

$$\frac{dw}{dt}(t) = f_x(x(t), y(t))\, x'(t) + f_y(x(t), y(t))\, y'(t)$$

in other notation $\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$ or

$w = f(x, y)$

$$\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}}$$

$x = x(t)$ $y = y(t)$

TL = Thinking Land

Thm 2 If: $w = f(x, y, z)$ is diff
$x = x(t)$ and $y = y(t)$ and $z = z(t)$ are diff w.r.t. t
then $w = f(x(t), y(t), z(t))$ is diff. w.r.t t

and

$w = f(x, y, z)$ ← TL

$$\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}}$$

▶ Do Ex 1

Thm 3 If: $w = f(x, y, z)$ is diff.
$x = g(r, s)$ and $y = h(r, s)$ and $z = k(r, s)$ are diff.
then w has partial derivatives w.r.t. r and s

and

TL: $w = f(x, y, z)$

$$\boxed{\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}}$$

AND

$$\boxed{\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}}$$

Rmk If $w = f(x, y)$, can view $w = f(x, y, z)$ with w a constant function of z and apply Thm 3 to get Cor 4 (below). Cor 5 is similar, with $w = f(x)$.

Let's think out before peeking

Cor 4 If $w = f(x,y)$ and $x = g(r,s)$ and $y = h(r,s)$ are diff. then w has partial der. wrt r and s

and

TL: $w = f(x, y)$

$$\boxed{\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}} \quad \text{AND} \quad \boxed{\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}}$$

Cor 5 If $w = f(x)$ and $x = g(r,s)$ is diff. then w has partial der. wrt r and s

and

TL: $w = f(x)$

$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r} \quad \text{AND} \quad \frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}$$

▶ Do Ex 2

Implicit Differentiation

Thm If $F(x,y)$ is diff. and the equation $F(x,y) = 0$ defines a diff. function of x

(think of as: a diff. function $y = g(x)$)

do not need to find !!

then

$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y}}$$

at any point where $F_y \neq 0$.

Thm If $F(x,y,z)$ is diff. and the equation $F(x,y,z) = 0$ defines a diff. function of x and y

(think of as: a diff. function $z = g(x,y)$)

do not need to find !!

then

$$\boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{-F_y}{F_z}}$$

at any point where $F_z \neq 0$.

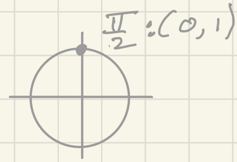
▶ Do Ex 3 and Ex 4

Ex 1 Thm 1 / Thm 2.

Let $W = x^2 y + y + xz$ and : $x = \cos\theta$, $y = \sin\theta$, $z = \theta^2$.

1.1 Express $\dfrac{dW}{d\theta}$ as a function of θ.

1.2 Evaluate $\left.\dfrac{dW}{d\theta}\right|_{\theta = \frac{\pi}{2}}$



Soln

1.1 Th

$$\frac{dW}{d\theta} =$$

Ex 2 Thm 3 (and: cor 4, Cor 5)

Let $w = x^4 y + y^2 z^3$ where

$$x = rse^{t}$$
$$y = rs^2 e^{-t}$$
$$z = r^2 s \sin t.$$

Evaluate $\frac{\partial w}{\partial s}$ when: $r = 2$, $s = 1$, $t = 0$.

Sol'n.

Implicit Differentiation Examples

Ex 3 Find $\frac{dy}{dx}$ if $x^3 + y^3 = 6xy$.

Soln. $x^3 + y^3 = 6xy \iff x^3 + y^3 - 6xy = 0$.

So let $F(x, y) =$ ______________________ .

Ex 4 Find $\frac{\partial z}{\partial x}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

Soln. $x^3 + y^3 + z^3 + 6xyz = 1 \iff x^3 + y^3 + z^3 + 6xyz - 1 = 0$.

So let $F(x, y, z) =$ ______________________