§ 14.3 Partial Derivatives

Warning $\partial \neq d$

partial ↗ ↖ derivative

<u>Def.</u> Let $f: D^2 \to \mathbb{R}^2$ w/ $N_\varepsilon((x_0, y_0)) \subset D^2 \subseteq \mathbb{R}^2$ for some $\varepsilon > 0$.

The (first-order) <u>partial derivatives</u> of f at (x_0, y_0)

1. w.r.t. x is $\left.\dfrac{\partial f}{\partial x}\right|_{(x_0, y_0)} \overset{\text{or}}{=} f_x(x_0, y_0) := \lim\limits_{h \to 0} \dfrac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$

2. w.r.t. y is $\left.\dfrac{\partial f}{\partial y}\right|_{(x_0, y_0)} \overset{\text{or}}{=} f_y(x_0, y_0) := \lim\limits_{h \to 0} \dfrac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$

<u>Rmk</u>

1. f_x is the rate of change of f as change x while holding y constant.
   To find f_x, differentiate $f(x,y)$ w.r.t. x and treat y as a constant.

2. f_y is the rate of change of f as change y while holding x constant.
   To find f_y, differentiate $f(x,y)$ w.r.t. y and treat x as a constant.

will use in Desmos 14.3.1 : Partial Derv.

14.3.2

Ex 1. Let

$$f(x,y) = \sqrt{9 - x^2 - y^2} \quad \underset{\text{friendly}}{\overset{\text{calculus}}{=}} \quad (9 - x^2 - y^2)^{1/2}$$

1.1. $\dfrac{\partial f}{\partial x}(x,y) =$

diff wrt x, treat as a constant : y.

1.2 $f_y(x,y) \overset{\text{notation}}{=} \dfrac{\partial f}{\partial y}(x,y) =$

diff wrt y, treat as a constant : x

1.3 $f_y(1,2) =$

Ex2 $\dfrac{\partial}{\partial z} z e^{xyz} =$

diff. wrt z, treat as constants : x, y.

Second-Order Partial Derivative.

N = notation.

Let $f : D^2 \to \mathbb{R}^2$ w/ $N_\varepsilon((x_0, y_0)) \subset D^2 \subseteq \mathbb{R}^2$ for some $\varepsilon > 0$.

Recall $f_x : D^2 \to \mathbb{R}^2$ with $f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f$.

and $f_y : D^2 \to \mathbb{R}^2$ with $f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f$.

Defs (and notation) for 2^{nd}-order

1. $f_{xx} \stackrel{\text{def}}{=} (f_x)_x = \frac{\partial f_x}{\partial x} = \frac{\partial}{\partial x} f_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$

2. $f_{yy} \stackrel{\text{def}}{=} (f_y)_y = \frac{\partial f_y}{\partial y} = \frac{\partial}{\partial y} f_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$

"mixed" partials bcs that partials wrt at least 2 different variables

3. $f_{xy} \stackrel{\text{def}}{=} (f_x)_y = \frac{\partial f_x}{\partial y} = \frac{\partial}{\partial y} f_x = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$

note "switch"

4. $f_{yx} \stackrel{\text{def}}{=} (f_y)_x = \frac{\partial f_y}{\partial x} = \frac{\partial}{\partial x} f_y = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$

order.
start wrt y
next wrt x

Rmk: Similarly for functions of 3-variables
Similarly for higher order.

3^{rd}-order

$$f_{xyz} \stackrel{\text{def}}{=} (f_{xy})_z = \frac{\partial}{\partial z} f_{xy} = \frac{\partial}{\partial z}(f_x)_y = \frac{\partial}{\partial z}\frac{\partial}{\partial y}\frac{\partial}{\partial x} f = \frac{\partial^3 f}{\partial z\,\partial y\,\partial x}$$

or

$$= (f_x)_{yz} = \frac{\partial}{\partial z}\frac{\partial}{\partial y} f_x = \frac{\partial}{\partial z}\frac{\partial}{\partial y}\frac{\partial}{\partial x} f = \frac{\partial^3 f}{\partial z\,\partial y\,\partial x}$$

4^{th}-order

$$f_{yyxx} \stackrel{\text{notation}}{=\!=} \frac{\partial^4 f}{\partial x^2\,\partial y^2}$$

14.3.4

Ex 3 Find all the second partial derivatives of

$$f(x,y) = y^3 + 5y^2 e^{4x} - \cos(x^2)$$

First need the 1st order:

- $f_x(x,y) = 0 + 5y^2 \, e^{4x}(4) - [-\sin(x^2)][2x] =$

$\Downarrow$

$f_x(x,y) = 20y^2 e^{4x} + 2x\sin x^2$

- $f_y(x,y) = 3y^2 + 10y\, e^{4x}$

$\leftarrow$ 1st order

Now can do the 2nd order:

not mixed

- $f_{xx}(x,y) =$
- $f_{yy}(x,y) =$

Mixed

- $f_{xy}(x,y) \overset{\text{i.e.}}{=} \frac{\partial}{\partial y} f_x(x,y) =$
- $f_{yx}(x,y) \overset{\text{i.e.}}{=} \frac{\partial f_y}{\partial x}(x,y) =$

Theorem. Mixed Partials / Clairaut: ↩ French ~ 1750

Let $f: D^2 \to \mathbb{R}$ w/ $D^2 \subseteq \mathbb{R}^2$ and D^2 is open

Let f_{xy} and f_{yx} be continuous on. Then for all $(x_0, y_0) \in D^2$

$$f_{xy}(x_0,y_0) = f_{yx}(x_0,y_0) \qquad (2^{nd})$$

In general [wiki]

Let $g: D^n \to \mathbb{R}$ w/ $D^n \subseteq \mathbb{R}^n$ and D open. Let $k \in \mathbb{N}$

Let all mixed partial of order k are cont. on U.

Then all mixed " are equal on U.

[I made up]

Ex Find f_{zyxx} for $f: \mathbb{R}^3 \to \mathbb{R}$ w/

$$f(x,y,z) = x\,(z^2 + ye^z)^3$$

Soln

$$f_{zyxx} = \frac{\partial}{\partial x}\frac{\partial}{\partial x}\underbrace{\frac{\partial}{\partial y}\underbrace{\frac{\partial}{\partial z} f}_{\text{yucky}}}_{\text{yuckier}} = \frac{\partial f^4}{\partial^2 x\,\partial y\,\partial z}$$

All mixed partials of order 4 are cont. So mixed partial Thm says can do partials in any order

pick an easier order

Geometric Interpretation of partial derivatives

- See Desmos 14.3.1 - goes with Ex 1.
- Start w/ a surface

$$\mathcal{S} \text{ is } z = f(x, y)$$

- Fix a point

$$P = (x_0, y_0, f(x_0, y_0)) \text{ on } \mathcal{S}.$$

- To examine

$$f_y(x_0, y_0)$$

fix $x = x_0$. So have a plane

$$\mathcal{P} \text{ is } x = x_0. \leftarrow \text{is parallel to } yz\text{-plane.}$$

- Consider the curve $\mathcal{C}$ in plane $\mathcal{P}$ and on $\mathcal{S}$, i.e.

$$\mathcal{C} := \mathcal{P} \cap \mathcal{S}$$

- <u>Staying in</u> the plane $\mathcal{P}$: $x = x_0$ ← is parallel to yz-plane
  the tangent line $\mathcal{L}$ to $\mathcal{C}$
  - at the point $(x_0, y_0, f(x_0, y_0))$
  - in the plane $x = x_0$ (so are $\parallel$ yz-plane)

  will have the equation

$$z - f(x_0, y_0) = m\,(y - y_0)$$

where $m = \dfrac{\Delta z}{\Delta y}$

So

(*) $f_y(x_0, y_0)$ = slope of tang. line at (x_0, y_0) to the curve $\mathcal{S} \cap \overbrace{[x = x_0]}^{\text{keeping } x \text{ fixed so is plane } \parallel yz \text{ plane}}$

(*) $f_x(x_0, y_0)$ = " $\mathcal{S} \cap \underbrace{[y = y_0]}_{\text{keeping } y \text{ fixed so is a plane } \parallel zx\text{-plane}}$.

$\boxed{\text{This page Set-up.}}$

Have $f : D^2 \to \mathbb{R}$ with $(x_0, y_0) \in D^2 \subseteq \mathbb{R}^2$ and D^2 is open.

Let $\Delta x = x - x_0$ and $\Delta y = y - y_0$ and

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

Def (some linear algebra would help now.....)

Let f_x and f_y exist at (x_0, y_0). Let

• Then f is differentiable at (x_0, y_0) provided

$$\lim_{(x,y) \to (x_0, y_0)} \frac{\| f(x,y) - [f(x_0, y_0) + (f_x(x_0, y_0)) \Delta x + (f_y(x_0, y_0))(\Delta y)] \|}{\| \langle \Delta x, \Delta y \rangle \|} = 0.$$

• f is differentiable $\iff$ f is differentiable at each $(x,y) \in D^2$.

• The graph of f is a smooth surface (in $\mathbb{R}^3$) $\iff$ f is differentiable at each $(x,y) \in D^2$,

Theorem 3 (p 828)

Let f_x and f_y be continuous at $(x_0, y_0) \in D^2$. Then

$$\Delta z = \underbrace{\left[(f_x(x_0, y_0))(\Delta x) + (f_y(x_0, y_0))(\Delta y) \right]}_{\text{approximates } \Delta z} + \underbrace{\varepsilon_1 \Delta x + \varepsilon_2 \Delta y}_{\text{"error"}}$$

with an

where $\lim_{\Delta x \to 0} \varepsilon_1 = 0$ and $\lim_{\Delta y \to 0} \varepsilon_2 = 0$.

Corollary f is differentiable at $(x_0, y_0) \Rightarrow f$ continous on D.

Remark f cont. $\not\Rightarrow$ f diff. but do have ....

Corollary f_x and f_y continuous on $D \Rightarrow f$ differentiable on D