

§ 14.3: Partial Derivatives

HO 14.3.1

Warning. partial derivative symbol $\partial \neq d$ for derivative

In Def 1-3, (x_0, y_0) is in the interior of $D^2 \subseteq \mathbb{R}^2$ and $f: D^2 \rightarrow \mathbb{R}$

Def 1 The (first-order) partial derivatives of f at the point (x_0, y_0)

• w.r.t. x is $\frac{\partial f}{\partial x}(x_0, y_0) \stackrel{\text{def}}{=} f_x(x_0, y_0) \stackrel{\text{def}}{=} \lim_{h \rightarrow 0} \frac{f(\overset{\text{fix}}{x_0+h}, \overset{\text{fix}}{y_0}) - f(x_0, y_0)}{h}$

while

• w.r.t. y is $\frac{\partial f}{\partial y}(x_0, y_0) \stackrel{\text{def}}{=} f_y(x_0, y_0) \stackrel{\text{def}}{=} \lim_{h \rightarrow 0} \frac{f(\overset{\text{fix}}{x_0}, \overset{\text{fix}}{y_0+h}) - f(x_0, y_0)}{h}$

Remarks to Def 1.

- $f_x(x_0, y_0)$ is the rate of change of f w.r.t. x when y is the fixed value y_0
 $f_y(x_0, y_0)$ is the rate of change of f w.r.t. y when x is the fixed value x_0
- $f_x(x_0, y_0)$ is the rate of change of f at (x_0, y_0) in the direction of $\hat{i}$.
 $f_y(x_0, y_0)$ is the rate of change of f at (x_0, y_0) in the direction of $\hat{j}$.
- To find f_x , differentiate $f(x, y)$ w.r.t. x and treat y as a constant.
 To find f_y , differentiate $f(x, y)$ w.r.t. y and treat x as a constant.

Def 2 Second-order partial derivatives other notation

• $\frac{\partial^2 f}{\partial y \partial x} \stackrel{\text{def}}{=} \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial y} f_x = \frac{\partial f_x}{\partial y} = (f_x)_y = f_{xy}$

• $\frac{\partial^2 f}{\partial x \partial y} \stackrel{\text{def}}{=} \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} f_y = \frac{\partial f_y}{\partial x} = (f_y)_x = f_{yx}$

• $\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial x \partial x} \stackrel{\text{def}}{=} \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} f_x = \frac{\partial f_x}{\partial x} = (f_x)_x = f_{xx}$

• $\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial y \partial y} \stackrel{\text{def}}{=} \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} f_y = \frac{\partial f_y}{\partial y} = (f_y)_y = f_{yy}$

mixed partials
bes
differentiate wrt
different variables

not mixed partials

Def 3 Similarly for higher-orders and functions of 3 variables, eg

• 4th-order: $\frac{\partial^4 f}{\partial x^2 \partial y \partial z} = \frac{\partial^4 f}{\partial x \partial x \partial y \partial z} \stackrel{\text{def}}{=} \left(\frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} \left(\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial z} \right) \right) \right) \right) \stackrel{\text{def}}{=} f_{zyxx}$

Let $f: D^2 \rightarrow \mathbb{R}$ with $(x_0, y_0) \in D^2 \subseteq \mathbb{R}^2$ and D^2 is open.

Theorem 4, Mixed Partial's Theorem Clairaut (French ~ 1750)

Let f_{xy} and f_{yx} be continuous on D^2 . Then

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0) \quad (2^{\text{nd}})$$

FYI In general (apply Thm 4 repeatedly).

Let $g: D^n \rightarrow \mathbb{R}$ w/ $D^n \subseteq \mathbb{R}^n$ and D^n open. Let $k \in \mathbb{N}$.

Let all mixed partials of order k are cont. on D^n .

Then all mixed partials of order k are equal on D^n .

Def 5. (some linear algebra would help now....)

Let f_x and f_y exist at (x_0, y_0) .

5.1 f is differentiable at (x_0, y_0) provided

$$\lim_{(x,y) \rightarrow (x_0, y_0)} \frac{\|f(x,y) - [f(x_0, y_0) + (f_x(x_0, y_0))(x-x_0) + (f_y(x_0, y_0))(y-y_0)]\|}{\| \langle x-x_0, y-y_0 \rangle \|} = 0.$$

5.2 f is differentiable $\Leftrightarrow$ f is differentiable at each $(x,y) \in D^2$.

5.3 The graph of f is a smooth surface (in $\mathbb{R}^3$) $\Leftrightarrow$ f is differentiable.

Corollary 6. f is differentiable on $D^2 \Rightarrow f$ continuous on D^2 .

Remark f cont. $\nRightarrow$ f diff. ... but do have:

Corollary 7 f_x and f_y continuous on $D^2 \Rightarrow f$ differentiable on D^2