§14.3: Partial Derivatives

HO 14.3.1

Warning. partial derivative symbol $\partial \neq d$ for derivative

In Def 1–3, (x_0, y_0) is in the interior of $D^2 \subseteq \mathbb{R}^2$ and $\boxed{f : D^2 \to \mathbb{R}}$

Def 1 The (first-order) partial derivatives of f at the point (x_0, y_0)

- w.r.t. x is $\frac{\partial f}{\partial x}(x_0,y_0) \overset{\text{or}}{=} f_x(x_0,y_0) \overset{\text{or}}{=} \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \overset{\text{def}}{:=} \lim\limits_{h\to 0} \frac{f(x_0+h, \overbrace{y_0) - f(x_0,}^{\text{fix}} y_0)}{h}$

while

- w.r.t. y is $\frac{\partial f}{\partial y}(x_0,y_0) \overset{\text{or}}{=} f_y(x_0,y_0) \overset{\text{or}}{=} \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \overset{\text{def}}{:=} \lim\limits_{h\to 0} \frac{f(\overbrace{x_0, y_0+h) - f(x_0}^{\text{fix}}, y_0)}{h}$

Remarks to Def 1.

1. $f_x(x_0,y_0)$ is the rate of change of f w.r.t. x when y is the fixed value y_0
   $f_y(x_0,y_0)$ is the rate of change of f w.r.t. y when x is the fixed value x_0
2. $f_x(x_0,y_0)$ is the rate of change of f at (x_0,y_0) in the direction of $\vec{\imath}$.
   $f_y(x_0,y_0)$ is the rate of change of f at (x_0,y_0) in the direction of $\vec{\jmath}$.
3. To find f_x, differentiate $f(x,y)$ w.r.t. x and treat y as a constant.
   To find f_y, differentiate $f(x,y)$ w.r.t. y and treat x as a constant.

Def 2 Second-order partial derivatives — other notation

mixed partials bcs differentiate wrt different variables:

- $$\frac{\partial^2 f}{\partial y \partial x} \overset{\text{def}}{:=} \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y} f_x = \frac{\partial f_x}{\partial y} = (f_x)_y = f_{xy}$$

first differentiate w.r.t. x

- $$\frac{\partial^2 f}{\partial x \partial y} := \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x} f_y = \frac{\partial f_y}{\partial x} = (f_y)_x = f_{yx}$$

- $$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial x \partial x} := \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x} f_x = \frac{\partial f_x}{\partial x} = (f_x)_x = f_{xx}$$

- $$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial y \partial y} := \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y} f_y = \frac{\partial f_y}{\partial y} = (f_y)_y = f_{yy}$$

Def 3 Similarly for higher-orders and functions of 3 variables, eg

- 4th-order: $$\frac{\partial^4 f}{\partial x^2 \partial y \partial z} = \frac{\partial^4 f}{\partial x \partial x \partial y \partial z} \overset{\text{def}}{:=} \left(\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right)\right)\right)\right) \overset{\text{so}}{=} f_{zyxx}$$

$\boxed{\text{This page Set-up.}}$

Let $f: D^2 \to \mathbb{R}$ with $(x_0, y_0) \in D^2 \subseteq \mathbb{R}^2$ and D^2 is open. Eg $D^2 = N_r(x_0, y_0)$.

Theorem 4. Mixed Partials Theorem Clairaut (French ~ 1750)

Let f_{xy} and f_{yx} be continuous on D^2. Then

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0) \qquad (2^{nd})$$

FYI In general (see Wiki)

Let $g: D^n \to \mathbb{R}$ w/ $D^n \subseteq \mathbb{R}^n$ and D^n open. Let $k \in \mathbb{N}$.

Let all mixed partials of order k are cont. on D^n.

Then all mixed partials of order k are equal on D^n.

Defs 5. (some linear algebra would help now.....)

Let f_x and f_y exist at (x_0, y_0).

5.1 f is differentiable at (x_0, y_0) provided

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\| f(x,y) - [f(x_0,y_0) + (f_x(x_0,y_0))(x-x_0) + (f_y(x_0,y_0))(y-y_0)] \|}{\| \langle x-x_0, y-y_0 \rangle \|} = 0.$$

5.2 f is differentiable $\iff$ f is differentiable at each $(x,y) \in D^2$.

5.3 The graph of f is a smooth surface (in $\mathbb{R}^3$) $\iff$ f is differentiable.

Corollary 6. f is differentiable on D^2 $\Rightarrow$ f continous on D^2.

Remark f cont. $\not\Rightarrow$ f diff. $\therefore$ but do have:

Corollary 7 f_x and f_y continuous on $D^2 \Rightarrow f$ differentiable on D^2