13.4 Curvature and Normal Vector – a practice example

Do the calculus and algebra <u>before</u> looking at the given solution.

<u>Ex.</u> Let

$$\vec{r}(t) = \langle e^t \cos t,\ e^t \sin t,\ e^t \rangle \quad \text{with } -\infty < t < \infty.$$

For $t \in \mathbb{R}$, find:

- unit tangent $\vec{T}(t)$,
- unit principle normal $\vec{N}(t)$,
- curvature $\kappa(t)$.

$\boxed{\text{Recap}}$ in abbreviated notation ⟨leaving out (t)⟩

- <u>unit tangent vector</u> $\quad \vec{T} = \dfrac{\vec{v}}{\|\vec{v}\|}$ is tangent to $\mathcal{C}$.
- <u>unit principle normal vector</u> $\quad \vec{N} = \dfrac{\vec{T}'}{\|\vec{T}'\|}$ $\quad \left[\begin{array}{l} \vec{T} \parallel \vec{v} \\ \vec{T} \perp \vec{N} \end{array}\right] \overset{\text{note}}{\Longrightarrow} \vec{N} \perp \vec{v}$
- <u>curvature</u> $\quad \kappa = \dfrac{\|\vec{T}'\|}{\|\vec{v}\|}$ give how much the curve is curving

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<u>Soln</u>

(1) $\vec{v}(t) = \vec{r}\,'(t) = \langle e^t \cos t + e^t(-\sin t),\ e^t \sin t + e^t \cos t,\ e^t \rangle$

$= e^t \langle \cos t - \sin t,\ \cos t + \sin t,\ 1 \rangle$

(2) $\|\vec{v}(t)\| = |e^t| \sqrt{\underbrace{(\cos t - \sin t)^2 + (\cos t + \sin t)^2}_{} + 1^2}$

$e^t > 0$ $\quad (\cos^2 t - \cancel{2\cos t \sin t} + \sin^2 t) + (\cos^2 t + \cancel{2\sin t \cos t} + \sin^2 t)$

$$= e^t \sqrt{2(\cos^2 t + \sin^2 t) + 1^2} = \sqrt{3}\, e^t$$

$$(3)\ \vec{T}(t) = \frac{\vec{v}(t)}{\|\vec{v}(t)\|} = \frac{e^t\langle \cos t - \sin t, \cos t + \sin t, 1\rangle}{\sqrt{3}\, e^t}$$

$$= \boxed{\frac{1}{\sqrt{3}}\langle \cos t - \sin t,\ \cos t + \sin t,\ 1\rangle = \vec{T}(t)}$$

$$(4)\ \vec{T}'(t) = \frac{1}{\sqrt{3}}\langle (-\sin t) - (\cos t),\ (-\sin t) + (\cos t),\ 0\rangle$$

$$= \frac{1}{\sqrt{3}}\langle -(\cos t + \sin t),\ \cos t - \sin t,\ 0\rangle$$

$$(5)\ \|\vec{T}'(t)\| = \frac{1}{\sqrt{3}}\sqrt{\underbrace{(\cos t + \sin t)^2 + (\cos t - \sin t)^2}_{\downarrow \text{ see calculation in } (2)} + 0^2}$$

$$= \frac{1}{\sqrt{3}}\sqrt{2 + 0^2} = \frac{\sqrt{2}}{\sqrt{3}}$$

$$(6)\ \vec{N}(t) = \frac{1}{\|\vec{T}'(t)\|}\vec{T}'(t) = \underbrace{\left(\frac{\sqrt{3}}{\sqrt{2}}\right)}_{(5)}\ \overbrace{\frac{1}{\sqrt{3}}\langle -(\cos t + \sin t), \cos t - \sin t, 0\rangle}^{(4)}$$

$$= \boxed{\frac{1}{\sqrt{2}}\langle -(\cos t + \sin t),\ \cos t - \sin t,\ 0\rangle = \vec{N}(t)}$$

$$(7)\ \kappa(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{v}(t)\|} = \underbrace{\frac{1}{\|\vec{v}(t)\|}}_{\downarrow (2)} \cdot \underbrace{\|\vec{T}'(t)\|}_{\downarrow (5)}$$

$$= \frac{1}{\sqrt{3}\, e^t} \cdot \frac{\sqrt{2}}{\sqrt{3}} = \boxed{\frac{\sqrt{2}}{3e^t} = \kappa(t)}$$

Source [Colley p202 Ex 11]