13.4 Curvature and Normal Vetor

Defs Let

- $\vec{r}: \mathbb{D}^1 \to \mathbb{R}^m$ be a smooth curve (e.g., position vector) ($\vec{r}\,'$ is cont. & never $\vec{0}$)
- $\vec{v} = \vec{r}\,'$
- $\mathcal{C}$ be the curve traced out by $\vec{r}$.

So the unit tangent vector-value function $T: \mathbb{D}^1 \to \mathbb{R}^m$ is

$$\vec{T}(t) := \frac{\vec{v}(t)}{\|\vec{v}(t)\|} \overset{\text{know}}{=} \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$$

is tangent to $\mathcal{C}$ at the ~~point~~ terminal point of the vector $\vec{r}(t)$

Def The unit principle normal vector-valued function $N: \mathbb{D}^1 \to \mathbb{R}^m$ is

$$\vec{N}(t) := \frac{\frac{d\vec{T}}{dt}(t)}{\left\|\frac{d\vec{T}}{dt}(t)\right\|}$$

Remark: $\vec{N}$ points towards the concave side of $\mathcal{C}$.

Note. The length $\vec{T}$ is constant $\overset{\S 13.1}{\Longrightarrow} \vec{T}(t) \cdot \vec{T}\,'(t) = 0 \Rightarrow \vec{T}(t) \cdot \vec{N}(t) = 0$.
So $\vec{T}(t) \perp \vec{N}(t)$. Since $\vec{T}(t) \parallel \vec{v}(t)$, get $\vec{N}(t) \perp \vec{v}(t)$.

Def The curvature function $\kappa: \mathbb{D}^1 \to [0, \infty)$ (Kappa)

$$\kappa(t) := \frac{1}{\|\vec{v}(t)\|} \left\| \frac{d\vec{T}}{dt}(t) \right\|$$

Remark: The curvature of a straight line is zero at each point.
Roughly speaking, the sharper a curve bends, the larger the curvature is.

Recap in (incorrect) abbreviated notation

- unit tangent vector $\vec{T} = \dfrac{\vec{v}}{\|\vec{v}\|}$ is tangent to $\mathcal{C}$.
- unit principle normal vector $\vec{N} = \dfrac{\vec{T}'}{\|\vec{T}'\|}$

$\vec{T} \parallel \vec{v}$

$\vec{N} \perp \vec{T}$

$\vec{N} \perp \vec{v}$

- curvature $\kappa = \dfrac{\|\vec{T}'\|}{\|\vec{v}\|}$ give how much the curve is curving

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Ex. Let $\vec{r}(t) = \langle e^t \cos t, e^t \sin t, e^t \rangle$ with $-\infty < t < \infty$.
For $t \in \mathbb{R}$, find: unit tangent $\vec{T}(t)$, unit principle normal $\vec{N}(t)$, curvature $\kappa(t)$

Soln

(1) $\vec{v}(t) = \vec{r}'(t) = \langle e^t \cos t + e^t(-\sin t),\ e^t \sin t + e^t \cos t,\ e^t \rangle$

$= e^t \langle \cos t - \sin t,\ \cos t + \sin t,\ 1 \rangle$

(2) $\|\vec{v}(t)\| = |e^t| \sqrt{\underbrace{(\cos t - \sin t)^2 + (\cos t + \sin t)^2}_{} + 1^2}$

$e^t > 0$ $\quad (\cos^2 t - \cancel{2\cos t \sin t} + \sin^2 t) + (\cos^2 t + \cancel{2 \sin t \cos t} + \sin^2 t)$

$= e^t \sqrt{2(\cos^2 t + \sin^2 t) + 1^2} = \sqrt{3}\, e^t$

(3) $\vec{T}(t) = \dfrac{\vec{v}(t)}{\|\vec{v}(t)\|} = \dfrac{e^t \langle \cos t - \sin t, \cos t + \sin t, 1 \rangle}{\sqrt{3}\, e^t}$

$= \boxed{\dfrac{1}{\sqrt{3}} \langle \cos t - \sin t,\ \cos t + \sin t,\ 1 \rangle = \vec{T}(t)}$

(4) $\vec{T}'(t) = \frac{1}{\sqrt{3}} \langle (-\sin t) - (\cos t),\ (-\sin t) + (\cos t),\ 0 \rangle$

$= \frac{1}{\sqrt{3}} \langle -(\cos t + \sin t),\ \cos t - \sin t,\ 0 \rangle$

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(1) $\vec{v}(t) = e^t \langle \cos t - \sin t, \cos t + \sin t, 1 \rangle$

(2) $\|\vec{v}(t)\| = \sqrt{3}\, e^t$

(3) $\vec{T}(t) = \frac{1}{\sqrt{3}} \langle \cos t - \sin t, \cos t + \sin t, 1 \rangle$

(4) $\vec{T}'(t) = \frac{1}{\sqrt{3}} \langle -(\cos t + \sin t), \cos t - \sin t, 0 \rangle$

(5) $\|\vec{T}'(t)\| = \frac{1}{\sqrt{3}} \sqrt{\underbrace{(\cos t + \sin t)^2 + (\cos t - \sin t)^2}_{\downarrow \text{ see calculation in (2)}} + 0^2}$

$$= \frac{1}{\sqrt{3}} \sqrt{2 + 0^2} = \frac{\sqrt{2}}{\sqrt{3}}$$

(6) $\vec{N}(t) = \frac{1}{\|\vec{T}'(t)\|} \vec{T}'(t) = \underbrace{\left(\frac{\sqrt{3}}{\sqrt{2}}\right)}_{(5)} \overbrace{\frac{1}{\sqrt{3}} \langle -(\cos t + \sin t), \cos t - \sin t, 0 \rangle}^{(4)}$

$$= \boxed{\frac{1}{\sqrt{2}} \langle -(\cos t + \sin t), \cos t - \sin t, 0 \rangle = \vec{N}(t)}$$

(7) $\kappa(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{v}(t)\|} = \underbrace{\frac{1}{\|\vec{v}(t)\|}}_{\downarrow (2)} \cdot \underbrace{\|\vec{T}'(t)\|}_{\downarrow (5)}$

$$= \frac{1}{\sqrt{3}\, e^t} \cdot \frac{\sqrt{2}}{\sqrt{3}} = \boxed{\frac{\sqrt{2}}{3e^t} = \kappa(t)}$$