§13.3 Arc Length (AL)

B.3 Handout

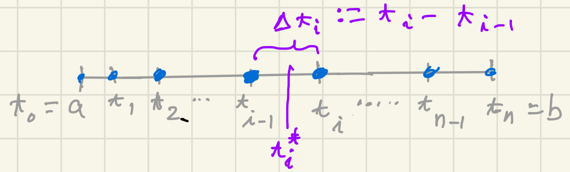
- Given a curve $\mathcal{C}$ parameterized by $\vec{r} : [a, b] \to \mathbb{V}^n$ ← $n = 2$ or 3

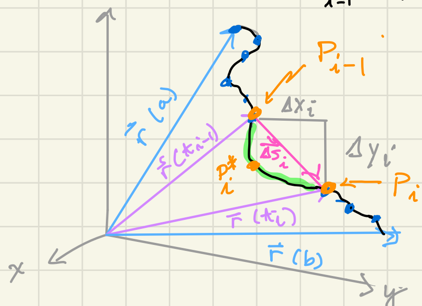
$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

- Goal: find the arc length AL of $\mathcal{C}$.

Rmk Will use this approach often.

Step 1 Partition the interval $[a, b]$. Make a seletion t_i^* where $t_{i-1} \le t_i^* \le t_i$.





Given $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$.

Let $P_j = (x(t_j), y(t_j), z(t_j))$.

Let $P_j^* = (x(t_j^*), y(t_j^*), z(t_j^*))$.

Step 2 Approx. AL as a sum of "typical elements"

$$AL = \sum_{i=1}^{n} (\text{AL of part of } \mathcal{C} \text{ btw } P_{i-1} \text{ and } P_i) \approx \sum_i \underbrace{\|\vec{\Delta S_i}\|}_{\text{typical elements}}$$

Step 3 Approx. a typical element. We got $\vec{\Delta S}_i \approx \underbrace{\vec{r}'(t_i^*)}_{\text{a vector}} \underbrace{\Delta t_i}_{\text{a scalar}}$

$$\|\vec{\Delta S}_i\| \approx \underbrace{\|\vec{r}'(t_i^*)\| \, \Delta t_i}_{\text{a scalar}}$$

Step 4 Step 2 and Step 3 $\Rightarrow$ $AL \approx \sum_i \|\vec{r}'(t_i^*)\| \Delta t$

Step 5 Take the limit as $\Delta t \to 0$ (so taking more and more t_i's) to get

$$AL = \lim_{\Delta t \to 0} \sum_i \|\vec{\Delta S_i}\| = \lim_{\Delta t \to 0} \sum_i \|\vec{r}'(t_i^*)\| \Delta t = \int_{t=a}^{t=b} \|\vec{r}'(t)\| \, dt$$

Take-Aways

Helpful notation:

- $d\vec{s} := \vec{r}'(t) \, dt$
- $ds \underset{\text{of } ds}{\overset{\text{think}}{=}}$ "$\| d\vec{s} \|$" $:= \underbrace{\|\vec{r}'(t)\| \, dt}_{\text{"integrating factor"}}$

Recall we had:

$$d\vec{s} \approx \vec{\Delta S}_i \approx \vec{r}'(t_i^*) \, \Delta t_i$$

$$ds \approx \|\vec{\Delta S}_i\| \approx \|\vec{r}'(t_i^*)\| \, \Delta t_i$$