§13.3 Arc Length (AL)

- Given a curve $\mathscr{C}$ parameterized by

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle, \quad a \le t \le b.$$

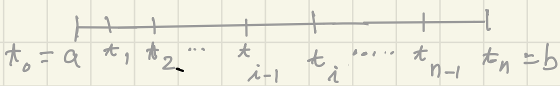
- Goal: Find the length of $\mathscr{C}$,
  ↳ called "arc length" (AL).

- Game Plan: express the Arc Length as an integral w.r.t. t.
  So want

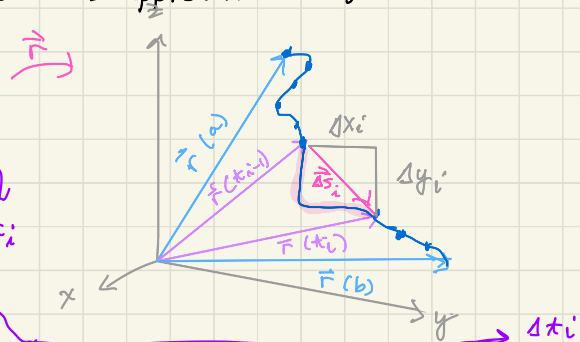
$$AL = \int_a^b \underbrace{(\text{some mess involving } t)}_{\text{a.k.a. (a function of } t): [a,b] \to \mathbb{R}^1} \, dt$$

- Think/recall Riemann Sums $\int_a^b f(t)\,dt \approx \sum_{i=1}^{n} f(t_i)\,\Delta t_i$

(*) An intuitive approach (will use this approach lots).



$$\underbrace{\quad}_{\Delta t_i := t_i - t_{i-1}}$$



$$AL \approx \sum_{i=1}^{n} \underbrace{\|\vec{\Delta s}_i\|}_{\boxed{\text{Typical element}}} \approx \sum_{i=1}^{n} (\text{mess})\,\Delta t_i$$

$$\vec{\Delta s}_i \overset{\text{def}}{:=} \vec{r}(t_i) - \vec{r}(t_{i-1}) = \boxed{\frac{\vec{r}(t_i) - \vec{r}(t_{i-1})}{t_i - t_{i-1}}} \; \boxed{(t_i - t_{i-1})}$$

$$\approx \boxed{\vec{r}\,'(t_i)} \; \boxed{\Delta t_i}$$

So have

$$\vec{\Delta s}_i \approx \overbrace{\vec{r}\,'(t_i)}^{\text{a vector}} \; \overbrace{\Delta t_i}^{\text{a scalar}}$$

$$\|\vec{\Delta s}_i\| \approx \|r'(t_i)\| \, \Delta t_i$$

To get an approximation of AL of whole cure add up the lengths of all the small segments, i.e., add up all the $\|\Delta s_i\|$.

$$AL \approx \sum_{i=1}^{n} \|\overrightarrow{\Delta s_i}\|$$

$$\approx \sum_{i=1}^{n} \|r'(t_i)\| \, \Delta t_i$$ ← from dividing $[a,b]$ into $\underline{\underline{n}}$ pieces.

Next, think of dividing $[a,b]$ into smaller & smaller pieces, i.e. let $n \to \infty$, to get

$$AL = \int_a^b \|r'(t)\| \, dt$$

Def. The arc length of a smooth curve $\mathcal{C}$ parameterized by the path $\vec{r} : [a,b] \to \mathbb{R}^3$ that is traced exactly once as t goes from a to b, is

$$AL = \int_a^b \|\vec{r}\,'(t)\| \, dt$$

p.s.: $\vec{r}$ smooth means $\frac{d\vec{r}}{dt}$ is continuous and never $\vec{0}$.

Helpful notation:

Recall we had:

- $d\vec{s} := \vec{r}\,'(t) \, dt$ ← $\overrightarrow{\Delta s_i} \approx \vec{r}\,'(t_i) \, \Delta t_i$
- $ds \overset{\text{think}}{\underset{\text{of as}}{=}}$ "$\|d\vec{s}\|$" ← $\|\overrightarrow{\Delta s_i}\| \approx \|r'(t_i)\| \, \Delta t_i$
- $ds := \underbrace{\|\vec{r}\,'(t)\| \, dt}_{\text{"integrating factor"}}$
- $\boxed{AL = \int_a^b \|\vec{r}\,'(t)\| \, dt = \int_a^b ds}$

Ex Find the arc length (AL) of the helix

$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ for $0 \le t \le \pi$.

Soln. If you want, for picture see Desmos 13.1.1 helix.

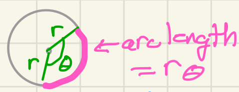
$$AL = \int_a^b \|\vec{r}\,'(t)\| \, dt$$

$$= \int_0^\pi \| \langle -\sin t, \cos t, 1 \rangle \| \, dt$$

$$= \int_0^\pi \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} \, dt$$

$$= \int_0^\pi \sqrt{\sin^2 t + \cos^2 t + 1} \, dt$$

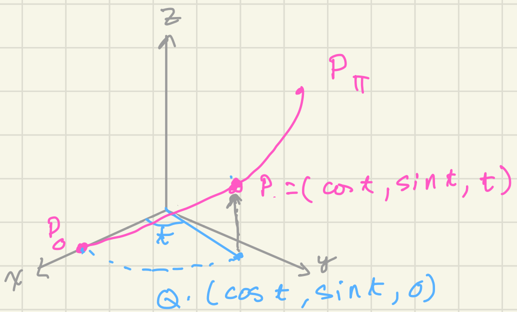
$$= \int_0^\pi \sqrt{1+1} \, dt = \sqrt{2} \int_0^\pi dt = \boxed{\sqrt{2}\,\pi}$$



Hey - neat problem - let's dive into it a bit.

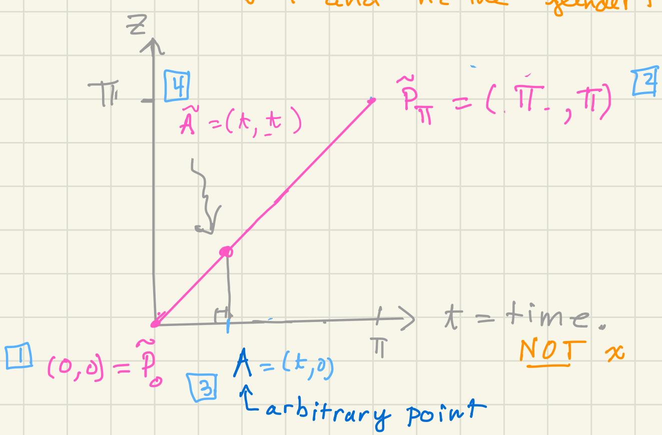
Let P_t = point on curve at time t. so $P_t = (\cos t, \sin t, t)$

So $P_0 = (1, 0, 0)$ and $P_\pi = (-1, 0, \pi)$



Next:

- Think of the helix on the cylinder $x^2+y^2=1$
- draw in the line $x=1$ (do!)
- cut the cylinder along the line $x=1$ and lie the cylinder flat.



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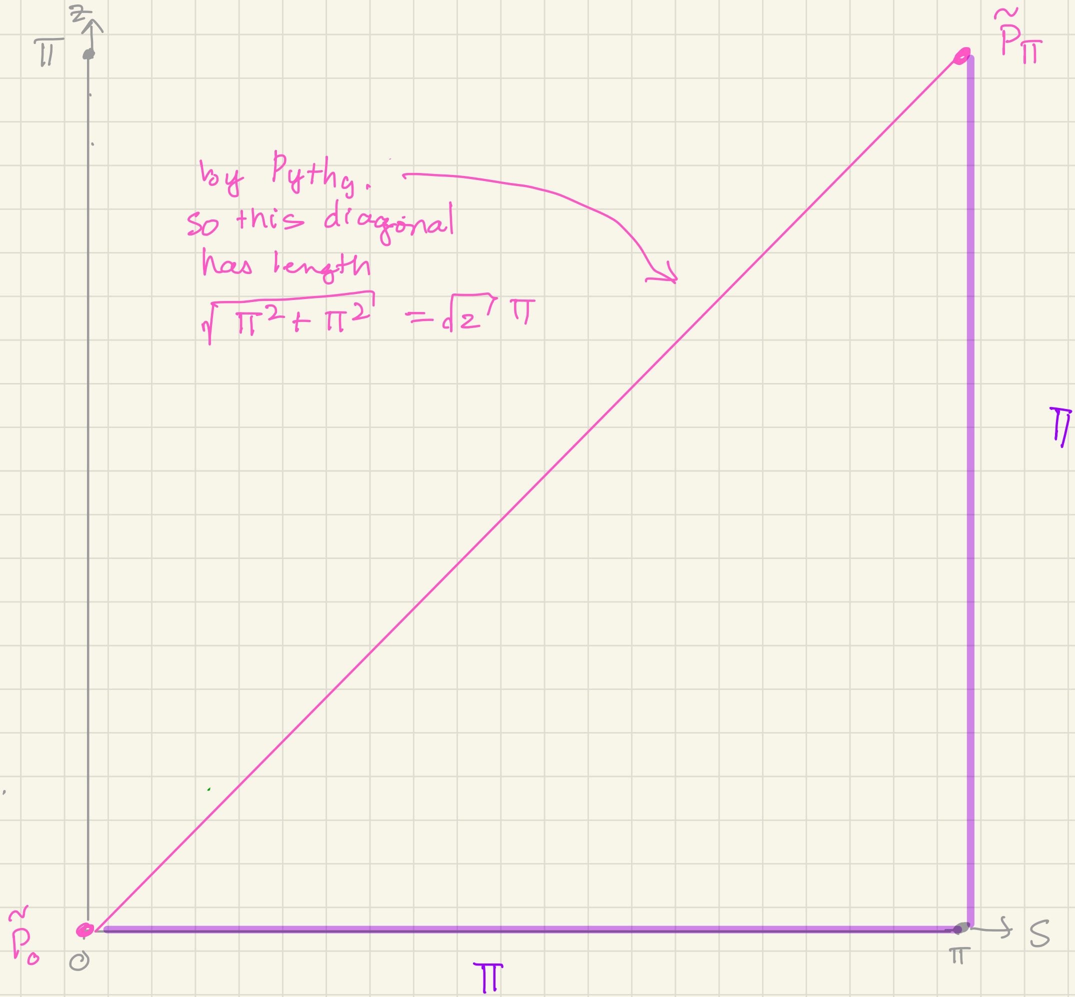
by Pythg.
so this diagonal
has length
$\sqrt{\pi^2+\pi^2} = \sqrt{2}\,\pi$



Recall :

Def. The <u>arc length</u> of a smooth curve $\mathscr{C}$ parameterized by the path $\vec{r} : [a,b] \to \mathbb{R}^3$ that is traced exactly once as t goes from a to b, is

$$AL = \int_a^b \|\vec{r}\,'(t)\| \, dt$$

Note

$$AL_{a \to b} = \int_a^b \|r'(\tau)\| \, d\tau$$ ← let $t = \tau$ ← "tau"

$$AL_{t_0 \to t} = \int_{t_0}^{t} \|r'(\tau)\| \, d\tau$$

<u>Def</u> The arc length parameter (with base point t_0) of a smooth curve $\mathscr{C}$ parameterized by the path $\vec{r}$ is

$$s(t) = \int_{t_0}^{t} \|r'(\tau)\| \, d\tau$$

Thus

$$s(t) = \begin{cases} \text{AL along } \mathscr{C} \text{ from time } t_0 \to t, & \text{when } t_0 < t \\ 0, & \text{when } t_0 = t \\ \text{negative the AL along } \mathscr{C} \text{ from time } t_0 \to t, & \text{when } t_0 > t \end{cases}$$

Recap from last few pages. Have done so far:

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Ex Find the arc length (AL) of the helix

$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ for $0 \le t \le \pi$.

Soln. If you want, for picture see Desmos 13.1.1 helix.

$$AL = \int_0^{\pi} \| \vec{r}\,'(t) \| \, dt = \int_0^{\pi} \sqrt{2} \, dt = \boxed{\sqrt{2}\,\pi}$$

Def The arc length parameter (with base point t_0) of a smooth curve $\mathscr{C}$ parameterized by the path $\vec{r}$ is

$$s(t) = \int_{t_0}^{t} \| r'(\tau) \| \, d\tau$$

Ex. Let $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$, $-\infty < t < \infty$.
Let $\mathscr{C}$ be curve $\vec{r}$ traces out.

- Find arc length parameter of $\mathscr{C}$, with base point $t_0 = 0$.

i.e. As a function of time t, express the arc length of the part of $\mathscr{C}$ traced out from time $t = 0$ to time $t > 0$.

Remark: your answer should not contain an integral.

Answer: $\boxed{s(t) =}$

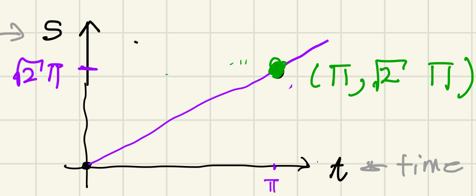
Soln

$$s(t) = \int_{t_0}^{t} \| r'(\tau) \| \, d\tau \overset{\text{from previous example.}}{=} \int_0^{t} \sqrt{2} \, d\tau = \sqrt{2}\,\tau \Big|_{\tau=0}^{\tau=t}$$

$$= \sqrt{2}\,t - \sqrt{2}\,(0) = \sqrt{2}\,t.$$

$\Rightarrow$ $s(t) = \sqrt{2}\,t$. ← go put in box!

Note $s(t) = \sqrt{2}\,t$ is a line thru pt $(0,0)$ w/ slope $= \sqrt{2}$.
and $s(\pi) = \sqrt{2}\,\pi$ (compare to last (i.e. above now) example).



This finishs 13.3.