§ 13.2 Integrals of paths.

Def. Let $\vec{F} : \mathbb{D}^1 \to \mathcal{V}^m$ ($m \in \mathbb{N}$, usually $m=2$ or $m=3$)
with $\vec{F}(t) = \langle f_1(t), f_2(t), \ldots, f_m(t) \rangle$
and each $f_i : \mathbb{D}^1 \to \mathbb{R}$ is integrable over $\mathbb{D}^1$
the interval $[a,b] \subseteq \mathbb{D}^1 \subseteq \mathbb{R}^1$.

1. Definite Integral

$$\int_a^b \vec{F}(t)\,dt = \left\langle \int_a^b f_1(t)\,dt, \int_a^b f_2(t)\,dt, \ldots, \int_a^b f_m(t)\,dt \right\rangle$$

2. Indefinite Integral

$$\int \vec{F}(t)\,dt = \vec{G}(t) + \vec{C}$$

provided

- $\vec{G} : \mathbb{D}^1 \to \mathcal{V}^m$
- $\vec{F} = \vec{G}'$ on $\mathbb{D}'$
- $\vec{C} \in \mathcal{V}^m$ is a constant vector.

Ex 1.1 $\int \langle 1, 2t \rangle\,dt = \langle \int 1\,dt, \int 2t\,dt \rangle$

$$= \langle t + c_1, t^2 + c_2 \rangle \overset{\text{i.e.}}{=} \langle t, t^2 \rangle + \langle c_1, c_2 \rangle$$

Ex 1.2 $\displaystyle\int_5^6 \langle 1, 2t \rangle\,dt = \langle t, t^2 \rangle \Big|_{t=5}^{t=6}$

$$= \langle 6, 36 \rangle - \langle 5, 25 \rangle = \langle 1, 11 \rangle$$

More on Motion

Know

so

$\frac{d}{dt}$ $\quad \vec{r}(t)$ position vector $\quad \int$ wrt t

$\vec{v}(t)$ velocity vector

$\frac{d}{dt}$ $\quad \vec{a}(t)$ acceleration vector $\quad \int$ wrt t

So, eg, $\int \vec{v}(t)\,dt = \vec{r}(t) + \vec{c}$.

Ex 2 A puffo is moving along a curve with this

(a) $\vec{a}(t) = \langle 0, \; 2, \; 9e^{3t} \rangle$

(b) $\vec{v}(0) = \langle 5, \; 0, \; 3 \rangle$

(c) $\vec{r}(1) = \langle 5, \; 8, \; e^3 \rangle$.

His position vector is (fill in the blanks)

$$\vec{r}(t) = \langle \_\_\_\_, \_\_\_\_, \_\_\_\_ \rangle$$

soln.

(1) $\vec{a}(t) = v'(t) \Rightarrow \int \vec{a}(t)\,dt = \vec{v}(t) + \vec{c}$ so . . .

$$v(t) = \int \langle 0, 2, 9e^{3t} \rangle \, dt = \langle c_1, \; 2t + c_2, \; 3e^{3t} + c_3 \rangle$$

(2) Find c_1, c_2, c_3 .... use the given (b)

given $\vec{v}(0) = \langle 5, 0, 3 \rangle$

(1) $\Rightarrow \vec{v}(0) = \langle c_1, \; 2(0) + c_2, \; 3e^{3(0)} + c_3 \rangle$

$= \langle c_1, \; c_2, \; 3 + c_3 \rangle$

$\xrightarrow[\text{coord.}]{\text{equate}}$ $c_1 = 5$, $c_2 = 0$, $c_3 = 0$.

so $\vec{v}(t) = \langle 5, \; 2t + 0, \; 3e^{3t} + 0 \rangle$

(3) $\vec{r}(t) = \int v(t)\,dt = \langle 5t + k_1, \; t^2 + k_2, \; e^{3t} + k_3 \rangle$

given $\vec{r}(1) = \langle 5, 8, e^3 \rangle$.

(3) $\Rightarrow \vec{r}(1) = \langle 5(1) + k_1, \; (1)^2 + k_2, \; e^{3(1)} + k_3 \rangle$ $\xrightarrow[\text{coord.}]{\text{equate}}$

$\hookrightarrow$ $5 = 5 + k_1$, $8 = 1 + k_2$, $e^3 = e^3 + k_3 \Rightarrow k_1 = 0, \; k_2 = 7, \; k_3 = 0$.

$\Rightarrow \vec{r}(t) = \langle 5t + 0, \; t^2 + 7, \; e^{3t} + 0 \rangle$

Ideal Projectile Motion

- Projectile Motion describes how an object fired at some angle from an intial position, and acted upon only by the force of gravity, moves in a vertical coordinate plane

- Let $\vec{v}_0$ = initial velocity. Let $v_0 = \|\vec{v}_0\|$.
  Let α be the angle btw $\vec{v}_0$ and the horizontal.
  Let g be gravity $= 32\ ft/sec^2 = 9.8\ m/sec^2$
  Note $\vec{r}(0)$ = initial point,

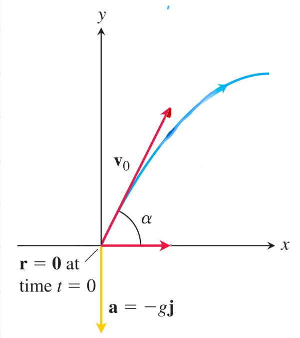

Figure 1

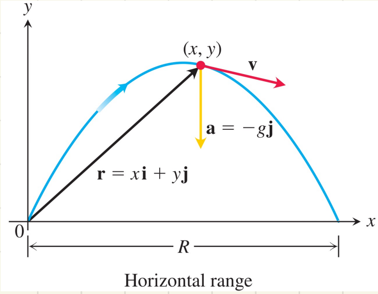

Figure 2

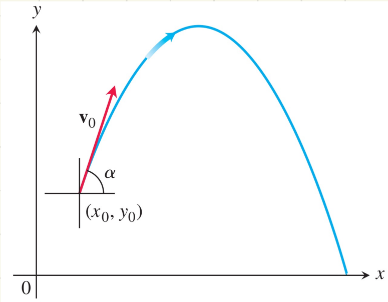

Figure 3

- Formulas (do not have to memorize but, if given, should be able use)

$\boxed{\text{Ideal Projectile}}$ $\quad \vec{r}(t) = \langle (v_0 \cos\alpha)t,\ (v_0 \sin\alpha)t - \frac{1}{2}gt^2 \rangle$

and when $\vec{r}(0) = (0,0)$ (i.e. launched from the origin)
over a horizontal surface (so like Figure 2 but NOT Figure 3)

$$\textit{Maximum height:} \qquad y_{max} = \frac{(v_0 \sin\alpha)^2}{2g}$$

$$\textit{Flight time:} \qquad t = \frac{2v_0 \sin\alpha}{g}$$

$$\textit{Range:} \qquad R = \frac{{v_0}^2}{g}\sin 2\alpha.$$

- Examples. Just apply formulas. See book. This finishes 13.2. Any questions.