Differentiation Rules for Vector-functions / paths

Given: differentiable $\vec{F} : D^1 \to \mathbb{R}^m$, $\vec{G} : D^1 \to \mathbb{R}^m$ } vector-valued

$f : D^1 \to \mathbb{R}^1$ ← scalar-valued

Let $\vec{K} : D^1 \to \mathbb{R}^n$ be a constant vector (so $\vec{c}\,(t_1) = \vec{c}\,(t_2)$ for each $t_1, t_2 \in D^1$)

$k \in \mathbb{R}$

Then (bcs diff. is coord-wise, the $\mathbb{R}$-rules extend to:)

1. $\frac{d}{dt}\left[\vec{K}\right] = \vec{0}$

2. $\frac{d}{dt}\left[\underbrace{f(t)}_{=k}\,\vec{F}(t)\right] = f'(t)\,\vec{F}(t) + f(t)\,\vec{F}'(t)$ product rule

2'. $\frac{d}{dt}\left[k\,\vec{F}(t)\right] = \boxed{\cancel{0\,\vec{F}(t)}} \to 0 + k\,\vec{F}'(t)$

3. $\frac{d}{dt}\left[\vec{F}(t) \pm \vec{G}(t)\right] = \vec{F}'(t) \pm \vec{G}'(t)$] (sum/difference)

4. $\frac{d}{dt}\left[\vec{F}(t) \cdot \vec{G}(t)\right] = \vec{F}'(t) \cdot \vec{G}(t) + \vec{F}(t) \cdot \vec{G}'(t)$ } dot & cross product rule

5. $\frac{d}{dt}\left[\vec{F}(t) \times \vec{G}(t)\right] = \vec{F}'(t) \times \vec{G}(t) + \vec{F}(t) \times \vec{G}'(t)$ }

6. $\frac{d}{dt}\left[\vec{F}(f(t))\right] = f'(t)\,\vec{F}'(f(t))$ chain rule

Motion

For a partical (aka puffo) flying thru space on a smooth curve $\mathscr{C}$ with position vector $\vec{r} : D^1 \to \mathbb{R}^3$.

1. $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ — position vector

2. $\vec{v}(t) = \frac{d}{dt}\vec{r}(t)$ — velocity vector.

3. $\vec{a}(t) = \frac{d}{dt}\vec{v}(t) = \frac{d^2}{dt^2}\vec{r}(t)$ — acceleration vector

4. $v(t) = \|\vec{v}(t)\|$ — speed (scalar)

$$\Downarrow$$

Warning: $\left\|\frac{d}{dt}\vec{r}(t)\right\| \neq \frac{d}{dt}\|\vec{r}(t)\|$

5. If $\vec{v}(t_0) \neq \vec{0}$, then $\vec{v}(t_0)$ is tangent to $\mathscr{C}$ at t_0 and is in the direction of motion.

$$\Downarrow$$

$\frac{\vec{v}(t_0)}{\|\vec{v}(t_0)\|}$ is the unit direct of motion at time t_0

6. $\vec{T}(t) = \frac{\vec{v}(t)}{\|\vec{v}(t)\|} \underset{\text{know}}{=} \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$ — unit tangent vector to $\mathscr{C}$ at the point $\vec{r}(t)$.

Ex If the length of $\vec{r}$ is constant, then $\vec{r} \cdot \frac{d\vec{r}}{dt} = 0$

$$\Downarrow$$

soln $|\vec{r}(t)| = c \Rightarrow c^2 = |\vec{r}(t)|^2 = \vec{r}(t) \cdot \vec{r}(t)$

$$\Rightarrow \frac{d}{dt}c^2 = \frac{d}{dt}\left[\vec{r}(t) \cdot \vec{r}(t)\right]$$

$$\Rightarrow 0 = \vec{r}\,'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}\,'(t)$$

$$\Rightarrow 0 = 2\,\vec{r}(t) \cdot \vec{r}\,'(t).$$

$$\Rightarrow \boxed{0 = r(t)\, r'(t)}$$

will use in 13.4