Ch 13: Vector valued Functions and Motion in Space

Notation

- $\mathbb{D}^1$ is an interval of $\mathbb{R}^1$. Eg: $[0,1]$ or $(-\infty,\infty) \stackrel{\text{i.e.}}{=} \mathbb{R}^1$ or $[0,2\pi)$.
- $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^m$. So have a point $\underbrace{(1,2,3)}_{\text{point}} \in \mathbb{R}^3$
  - set/collection of all points in $\mathbb{R}^m$.
- $\mathbb{V}^2$, $\mathbb{V}^3$, $\mathbb{V}^m$. So have a vector $\overbrace{\langle 1,2,3\rangle}^{\text{vector}} \in \mathbb{V}^3$
  - set/collection of all vectors in $\mathbb{R}^m$.

usually, m is 2 or 3.
2D 3D

Ch 13 considers functions $\vec{F} : \mathbb{D}^1 \longrightarrow \mathbb{V}^m$

domain $\mathbb{D}^1 \subseteq \mathbb{R}^1$ points in $\mathbb{R}^1$

vectors in $\mathbb{R}^m$

§13.1 Curves in space and their tangents.

Def A path, as called a vector-valued function, is a function

$$\vec{F} : \mathbb{D}^1 \to \mathbb{V}^m.$$

- When $m=3$, $\vec{F}$ has the form

$$\vec{F}(t) = \langle f_1(t), f_2(t), f_3(t) \rangle \quad \text{where } t \in \mathbb{D}^1$$

which can also be written as

$$\vec{F}(t) = \langle x(t), y(t), z(t) \rangle \quad \text{where } t \in \mathbb{D}^1.$$

- One thinks of the head/end-pt of $\vec{F}$ as tracing out (i.e. parameterizing) a (1-D) space curve $\mathcal{C}$ in $\mathbb{R}^m$.

Do Ex 1-6 on page 5.

Def. (Calculus I). For $f : \mathbb{D}^1 \to \mathbb{R}$ and $L \in \mathbb{R}$

$\lim_{t\to t_0} f(t) = L$ means

for each $\varepsilon > 0$, there is $\delta > 0$ satisfying

if $0 < |t - t_0| < \delta$ and $t \in \mathbb{D}^1$ then $|f(t) - L| < \varepsilon$.

Def 1 (Calculus III). For $\vec{F} : \mathbb{D}^1 \to \mathbb{V}^m$ and $\vec{L} \in \mathbb{V}^m$

$\vec{L} = \langle l_1, l_2, \dots, l_m \rangle$

$\lim_{t\to t_0} \vec{F}(t) = \vec{L}$ means

for each $\varepsilon > 0$, there is $\delta > 0$ satisfying

if $0 < |t - t_0| < \delta$ and $t \in \mathbb{D}^1$ then $\|\vec{F}(t) - \vec{L}\| < \varepsilon$.

Note bcs can work coordinate-wise

$$\lim_{t\to t_0} \vec{F}(t) = \vec{L} \overset{\text{def}}{\iff} \lim_{t\to t_0} \langle f_1(t), f_2(t), \dots f_m(t) \rangle = \langle l_1, l_2, \dots, l_m \rangle$$

$$\overset{\text{Note}}{\iff} \text{for each coordinate } j,\ 1 \le j \le m,\quad \lim_{t\to t_0} f_j(t) = l_j$$

Def 2a $\vec{F}$ is continuous at $t = t_0 \overset{\text{def}}{\iff} \lim_{t\to t_0} \vec{F}(t) = \vec{F}(t_0)$

$$\overset{\text{note}}{\iff} \text{for each coordinate } j,\quad \lim_{t\to t_0} f_j(t) = f_j(t_0)$$

Def 2b $\vec{F}$ is cont. on $\mathbb{D}^1 \iff \vec{F}$ is cont at each $t \in \mathbb{D}^1$.

Do Ex 7–9 on page 6.

Def 3a $\vec{F}$ is <u>differiable</u> at $t \iff \forall j$, f_j is diff. at t

in which case $\vec{F}' : \mathbb{D}^1 \to \mathbb{V}^m$ and

$$\boxed{\vec{F}'(t) = \frac{d\vec{F}}{dt} = \lim_{\Delta t\to 0} \frac{\vec{F}(t+\Delta t) - \vec{F}(t)}{\Delta t} = \left\langle \frac{df_1}{dt}, \frac{df_2}{dt}, \dots, \frac{df_m}{dt} \right\rangle}$$

just take derivates coordinatewise

Def 3b $\vec{F}$ is diff. on $\mathbb{D}^1 \iff \vec{F}$ is diff. at each $t \in \mathbb{D}^1$.

Def 4 (New)

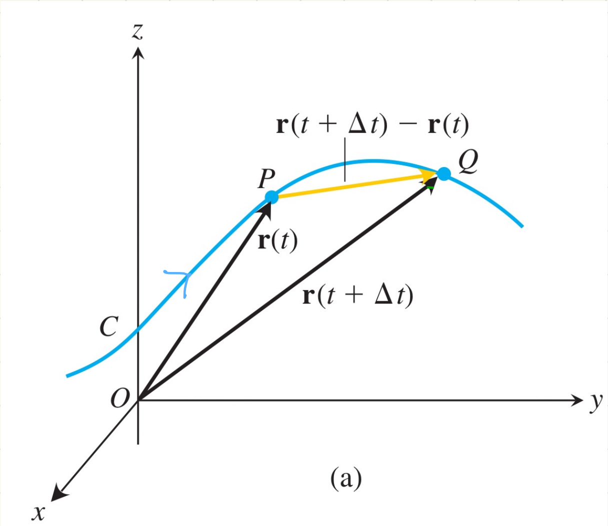
The curve $\mathcal{C}$ traced by $\vec{F}$ is <u>smooth</u> $\iff \frac{d\vec{F}}{dt}$ is cont. <u>and</u> never $\vec{0}$.

$\overset{\text{i.e.}}{\iff}$ "$\vec{F}$ has continuous non-vanishing derivative"
in short

(*) Fact 5 $\vec{F}'(t)$ is tangent to the curve $\mathscr{C}$ traced out by $\vec{F}$ at time (t).

so $\vec{F}$ is $\vec{r}$

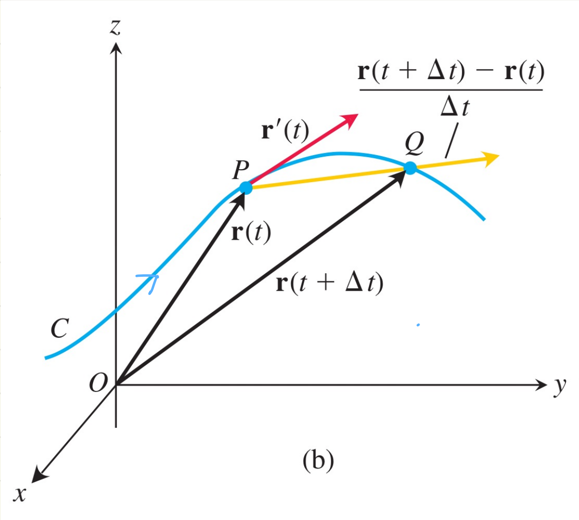
picture for the path $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$.
(from book page 763)



(a)

$\vec{r}'(t) \stackrel{\text{def 3a}}{=}$

$$\lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$



(b)

Now let $\Delta t \to 0$ and will get

- the point Q approaches the point P, along the curve C
- and in the limit, the vector $\dfrac{\vec{PQ}}{\Delta t}$ becomes the tangent vector $\vec{r}'(t)$

Motion – a typical application.

For a partical (aka puffo) flying thru space on a smooth curve $\mathscr{C}$ with position vector $\vec{r} : D^1 \to \mathbb{R}^3$.

1. $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ — position vector
2. $\vec{v}(t) = \frac{d}{dt}\vec{r}(t)$ — velocity vector.
3. $\vec{a}(t) = \frac{d}{dt}\vec{v}(t)$ — acceleration vector

4. $v(t) = \|\vec{v}(t)\|$ — speed (scalar) function

$\|$

Warning: $\|\frac{d}{dt}\vec{r}(t)\| \neq \frac{d}{dt}\|\vec{r}(t)\|$

5. If $\vec{v}(t_0) \neq \vec{0}$, then :
   - $\vec{v}(t_0)$ is tangent to $\mathscr{C}$ at t_0

   and

   - $\vec{v}(t_0)$ is in the direction of motion.

$\Downarrow$

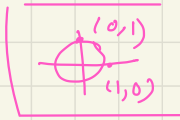
$\frac{\vec{v}(t_0)}{\|\vec{v}(t_0)\|}$ is the unit direction of the motion at time t_0

ps: useful for finding tangent lines to curve.

Do Ex 10 – 15, starting page 6

For Ex 1-5. Let a puffo have position vector

$$\boxed{\vec{r}(t) = \langle 4\sin t, 2\cos t\rangle}$$ with $0 \le t \le \pi/2$.



So $\vec{r} : [0, \pi/2] \to \mathbb{R}^2$ is a path.

Ex 1. What are the coordinate functions of $\vec{r}$?

$x(t) =$ ________ and $y(t) =$ ________

or sometimes just write

$x =$ ________ and $y =$ ________

Ex 2 $\vec{r}(0) = \langle$ ________ , ________ $\rangle$

$\vec{r}(0) = \langle \quad , \quad \rangle$

Ex 3 $\vec{r}(\frac{\pi}{2}) = \langle$ ________ , ________ $\rangle$

$\vec{r}(\frac{\pi}{2}) = \langle \quad , \quad \rangle$

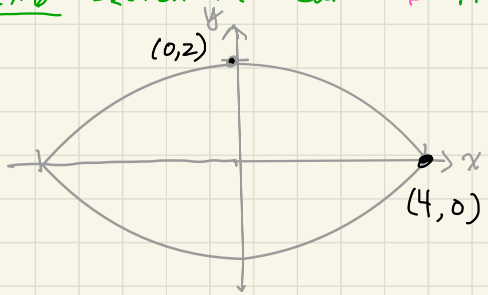
Ex 4. $\vec{r}(\pi/4) = \langle$ ________ , ________ $\rangle$

$\vec{r}(\pi/4) = \langle \quad , \quad \rangle =$

Ex 5 The coord. functions satisfy the equation ____________.

$$\left(\frac{4\sin t}{\square}\right)^2 + \left(\frac{2\cos t}{\square}\right)^2 = 1 \iff$$

Ex 6 Sketch the curve $\mathcal{C}$ that $\vec{r}$ traces out.



Have $\boxed{\vec{r}(t) = \langle 4\sin t, 2\cos t\rangle}$, $0 \le t \le \pi/2$.

Ex 7 $\lim_{t\to\pi/4} \vec{r}(t) = \langle$ ________, ________ $\rangle$

$\lim_{t\to\pi/4} \langle 4\sin t, 2\cos t\rangle =$

Ex 8 Is $\vec{r}$ continuous at $\pi/4$? ________________

Ex 9 Is $\vec{r}$ continuous on $[0, \pi/2]$? ________________

Ex 10 The puffo's velocity vector is $\vec{v}(t) = \langle$ ______, ______ $\rangle$

$\vec{v}(t) = \vec{r}\,'(t) = D_t \langle \quad , \quad \rangle =$

Ex 11 $\vec{v}(\frac{\pi}{4}) = \langle$ ________, ________ $\rangle$

$\vec{v}(\pi/4) = \langle \quad , \quad \rangle$

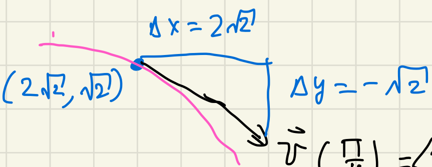
Ex 12 A parameterization of the tangent line $\mathcal{L}$ to the puffo's path when $t_0 = \pi/4$ is:

$x(t) = \quad + \quad t$

$y(t) =$ ______ $+$ ______ t

a point on $\mathcal{L}$, see Ex 4 — $\mathcal{L}$ is in same direction as $\vec{r}\,'(\frac{\pi}{4}) = \vec{v}(\pi/4)$

Ex 13 The tangent line (in Ex 12) is $y =$ ________________



$\vec{v}(\frac{\pi}{4}) = \langle 2\sqrt{2}, -\sqrt{2}\rangle$

$m = \frac{\Delta y}{\Delta x} = \frac{-\sqrt{2}}{2\sqrt{2}} = \frac{-1}{2}$

so

Ex 14 HW. Compare Ex 13 to MML HW over 11.1 + 11.2; 11.2.3 (think question 7)

Ex 15 So our puffo has:

position vector is: $\vec{r}(t) = \langle 4\sin t,\ 2\cos t \rangle$

velocity vector is: $\vec{v}(t) = \vec{r}\,'(t) = \langle 4\cos t,\ -2\sin t \rangle$

accelation vector is: $\vec{a}(t) =$ ______ $=$ ____________________

speed function is: $\|\vec{v}(t)\| =$ ____________________

End of Examples using $\vec{r}(t) = \langle 4\sin t,\ 2\cos t \rangle$

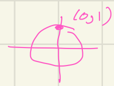
$\boxed{\text{You can now do MML HW for 13.1}}$

Ex 16 Our puffo is flying along the helix

$\vec{r}(t) = \langle \cos t,\ \sin t,\ t \rangle$ for $0 \le t \le 2\pi$

Find is velocity vector at $t = \pi/2$,

soln $\vec{v}(t) = \vec{r}\,'(t) = \langle$ ______, ______, ______ $\rangle$



$\vec{v}\left(\frac{\pi}{2}\right) = \langle$ ______, ______, ______ $\rangle =$

Ex 17 Ex 16 is illustrated in Desmos 13.1.1 so lets do this Desmos.
Let's see Desmos 13.1.1 then answer:

Question. What will happen when the Grinch turns off the magnetic switch at time $t = \pi/2$?

Ex 18 of when $\vec{r}$ is on (2D a circle / 3D a sphere) with Center = Origin

↑ will use in 13.4.

If the length of $\vec{r}$ is constant, then $\vec{r} \cdot \frac{d\vec{r}}{dt} = 0$

Soln. $\|\vec{r}(t)\| = c \Rightarrow c^2 = |\vec{r}(t)|^2 = \vec{r}(t) \cdot \vec{r}(t)$

$$\Rightarrow \frac{d}{dt} c^2 = \frac{d}{dt}[\vec{r}(t) \cdot \vec{r}(t)]$$

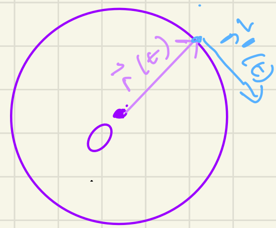
See below 4.

$$\Rightarrow 0 = \vec{r}\,'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}\,'(t)$$

$$\Rightarrow 0 = 2\vec{r}(t) \cdot \vec{r}\,'(t)$$

$$\Rightarrow \boxed{0 = \vec{r}(t) \cdot \vec{r}\,'(t)}$$

Recall, for circles "a tangent line is ⊥ to a radius"



Fact If $\vec{r}(t) \cdot \vec{r}\,'(t) = 0$ for all t, then $\|\vec{r}\|$ is constant. Bk Exercise 13.1.41.

Differentiation Rules for Vector-functions / paths

Given: differentiable $\vec{F} : \mathbb{D}^1 \to \mathbb{V}^m$, $\vec{G} : \mathbb{D}^1 \to \mathbb{V}^m$ } vector-valued

$f : \mathbb{D}^1 \to \mathbb{R}^1$ ← scalar-valued

Let $\vec{K} : \mathbb{D}^1 \to \mathbb{V}^m$ be a constant vector (eg $\vec{K}(t) = \langle 1, 2, 3\rangle$ for each t)

$k \in \mathbb{R}$

Then (bcs diff. is coord-wise, the $\mathbb{R}$-rules extend to!)

1. $\frac{d}{dt}[\vec{K}] = \vec{0}$

2. $\frac{d}{dt}[f(t)\vec{F}(t)] = f'(t)\vec{F}(t) + f(t)\vec{F}\,'(t)$ product rule

2'. $\frac{d}{dt}[\underbrace{k}_{=f(t)} \vec{F}(t)] = \boxed{\cancel{0\vec{F}(t)}}^{\,0} + k\vec{F}\,'(t)$ "constanst come out"

3. $\frac{d}{dt}[\vec{F}(t) \pm \vec{G}(t)] = \vec{F}\,'(t) \pm \vec{G}\,'(t)$] (sum/difference)

4. $\frac{d}{dt}[\vec{F}(t) \cdot \vec{G}(t)] = \vec{F}\,'(t) \cdot \vec{G}(t) + \vec{F}(t) \cdot \vec{G}\,'(t)$ } dot & cross product rule

5. $\frac{d}{dt}[\vec{F}(t) \times \vec{G}(t)] = \vec{F}\,'(t) \times \vec{G}(t) + \vec{F}(t) \times \vec{G}\,'(t)$

6. $\frac{d}{dt}[\vec{F}(f(t))] = f'(t)\,\vec{F}\,'(f(t))$ chain rule

This finishs 13.1. Any questions?