

Notation

- $\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^m$ • So have a point $(1, 2, 3) \in \mathbb{R}^3$
 - $\mathbb{V}^2, \mathbb{V}^3, \mathbb{V}^m$ • So have a vector $\langle 1, 2, 3 \rangle \in \mathbb{V}^3$
 - $\mathbb{D}^1$ is an interval of $\mathbb{R}^1$. Eg: $[0, 1]$ or $[0, 2\pi)$ or $(-\infty, \infty) \stackrel{\text{i.e.}}{=} \mathbb{R}^1$.
- Diagram: $\mathbb{D}^1$ (interval) $\xrightarrow{\text{domain in } \mathbb{R}^1}$ $\mathbb{F} : \mathbb{D}^1 \rightarrow \mathbb{V}^m$ (vector in $\mathbb{V}^m$)

Ch 13 considered vector-valued functions $\vec{F} : \mathbb{D}^1 \rightarrow \mathbb{V}^m$

§ 13.1 Curves in space and their tangents.

- A vector-value function $\vec{r} : \mathbb{D}^1 \rightarrow \mathbb{V}^3$ has the form

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle \text{ for } t \in \mathbb{D}^1$$

where $x : \mathbb{D}^1 \rightarrow \mathbb{R}^1$ and $y : \mathbb{D}^1 \rightarrow \mathbb{R}^1$ and $z : \mathbb{D}^1 \rightarrow \mathbb{R}^1$.
- We think of $\vec{r}(t)$ as a puffo's position vector.
 Then the ends/heads of $\vec{r}$ trace out the curve/path in $\mathbb{R}^3$
 $\{ \langle x(t), y(t), z(t) \rangle \in \mathbb{R}^3 : t \in \mathbb{D}^1 \}$, $\leftarrow$ a subset of $\mathbb{R}^3$.
 which is the puffo's path.
- Recall we did this already for when $\vec{r}$ was a line in $\mathbb{R}^3$.

Limits

Calc 1 Had $f : \mathbb{D}^1 \rightarrow \mathbb{R}^1$ and $L \in \mathbb{R}^1$ and $t_0 \in \mathbb{D}^1$.
 $\lim_{t \rightarrow t_0} f(t) = L \Leftrightarrow \lim_{t \rightarrow t_0} |f(t) - L| = 0$ (abs. value)
 a function $g : \mathbb{R}^1 \rightarrow \mathbb{R}^1$

Calc 3 Have $\vec{F} : \mathbb{D}^1 \rightarrow \mathbb{V}^m$ and $\vec{L} \in \mathbb{V}^m$ and $t_0 \in \mathbb{D}^1$, $m=2$ or 3 .
 So $\vec{F}(t) = \langle f_1(t), f_2(t), \dots, f_m(t) \rangle$ and $\vec{L} = \langle l_1, l_2, \dots, l_m \rangle$
 $\leftarrow f_i : \mathbb{D}^1 \rightarrow \mathbb{R}^1$ $\leftarrow l_i \in \mathbb{R}^1$

Def $\lim_{t \rightarrow t_0} \vec{F}(t) = \vec{L} \Leftrightarrow \lim_{t \rightarrow t_0} \|\vec{F}(t) - \vec{L}\| = 0$ ('lengths').
 a function $h : \mathbb{R}^1 \rightarrow \mathbb{R}^1$

< since calculate length > $\Leftrightarrow \lim_{t \rightarrow t_0} f_i(t) = l_i$ for each $i=1, 2, \dots, m$.
 coordinate-wise

Upside $\lim_{t \rightarrow t_0} \langle f_1(t), f_2(t) \rangle = \langle \lim_{t \rightarrow t_0} f_1(t), \lim_{t \rightarrow t_0} f_2(t) \rangle$
 provided each $\lim_{t \rightarrow t_0} f_i$ exists. Otherwise we say $\lim_{t \rightarrow t_0} \vec{F}(t)$ DNE.
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$$\vec{F}(t) = \langle f_1(t), f_2(t), \dots, f_m(t) \rangle \quad \boxed{\text{Continuity}}$$

Def $\vec{F}$ is continuous at $t=t_0 \iff \lim_{t \rightarrow t_0} \vec{F}(t) = \vec{F}(t_0)$

note $\iff \lim_{t \rightarrow t_0} f_i(t) = f_i(t_0)$ for each coordinate $i=1, \dots, m$.

Def $\vec{F}$ is continuous on $\mathbb{D}^1 \iff \vec{F}$ is cont. at each $t_0 \in \mathbb{D}^1$

Derivative

Def $\vec{F}$ is differentiable at $t \iff f_i$ is differentiable at t for each coord. $i=1, \dots, m$, in which case $\vec{F}' : \mathbb{D}^1 \rightarrow \mathbb{V}^m$ and.

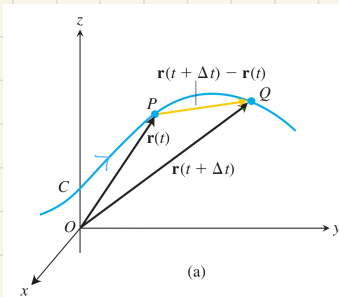
$$\vec{F}'(t) = \frac{d\vec{F}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\vec{F}(t+\Delta t) - \vec{F}(t)}{\Delta t} = \left\langle \frac{df_1}{dt}, \frac{df_2}{dt}, \dots, \frac{df_m}{dt} \right\rangle$$

Def $\vec{F}$ is diff. on $\mathbb{D}^1 \iff \vec{F}$ is diff. at each $t \in \mathbb{D}^1$

new Def The curve γ traced by $\vec{F}$ is smooth $\iff \frac{d\vec{F}}{dt}$ is cont. and never $\vec{0}$.

just take derivatives coordinate wise

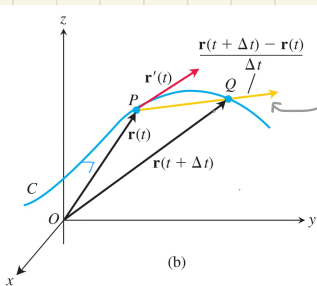
Fact $\vec{r}'(t)$ is tangent to the curve traced out by $\vec{r}$ at time t .



$$\vec{r}'(t) \stackrel{\text{def}}{=} \lim_{\Delta t \rightarrow 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

Now let $\Delta t \rightarrow 0$ and will get

- the point Q approaches the point P, along the curve C
- and in the limit, the vector



$$\frac{\vec{PQ}}{\Delta t}$$

becomes

the tangent vector $\vec{r}'(t)$

Motion

For a particle (aka puffo) flying thru space on a smooth curve γ with position vector $\vec{r} : \mathbb{D}^1 \rightarrow \mathbb{R}^3$.

1. $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

position vector

2. $\vec{v}(t) = \frac{d}{dt} \vec{r}(t)$

velocity vector.

3. $\vec{a}(t) = \frac{d}{dt} \vec{v}(t)$

acceleration vector

4. $v(t) = \|\vec{v}(t)\|$

speed (scalar) function

Warning: $\left\| \frac{d}{dt} \vec{r}(t) \right\| \neq \frac{d}{dt} \|\vec{r}(t)\|$

5. If $\vec{v}(t_0) \neq \vec{0}$, then:

- $\vec{v}(t_0)$ is tangent to γ at t_0

and

- $\vec{v}(t_0)$ is in the direction of motion.

↓

$\frac{\vec{v}(t_0)}{\|\vec{v}(t_0)\|}$ is the (unit) direction of the motion at time t_0

Differentiation Rules for Vector-functions, / paths

Given: differentiable $\vec{F}: D' \rightarrow \mathbb{V}^m$ } vector-valued
 $\vec{G}: D' \rightarrow \mathbb{V}^m$ }
 $f: D' \rightarrow \mathbb{R}^1$ } scalar-valued

Let $\vec{k}: D' \rightarrow \mathbb{V}^m$ be a constant vector (eg $\vec{k}(t) = \langle 1, 2, 3 \rangle$ for each t)
 $k \in \mathbb{R}$

Then (bes diff. in coord-wise, the $\mathbb{R}$ -rules extend to!)

1. $\frac{d}{dt} [\vec{k}] = \vec{0}$

2. $\frac{d}{dt} [f(t) \vec{F}(t)] = f'(t) \vec{F}(t) + f(t) \vec{F}'(t)$

product rule

↓
 2'. $\frac{d}{dt} [k \vec{F}(t)] = \boxed{0 \vec{F}(t)} + k \vec{F}'(t)$

"constant come out"

3. $\frac{d}{dt} [\vec{F}(t) \pm \vec{G}(t)] = \vec{F}'(t) \pm \vec{G}'(t)$

(sum/difference)

4. $\frac{d}{dt} [\vec{F}(t) \cdot \vec{G}(t)] = \vec{F}'(t) \cdot \vec{G}(t) + \vec{F}(t) \cdot \vec{G}'(t)$

5. $\frac{d}{dt} [\vec{F}(t) \times \vec{G}(t)] = \vec{F}'(t) \times \vec{G}(t) + \vec{F}(t) \times \vec{G}'(t)$

} dot & cross
product
rule

6. $\frac{d}{dt} [\vec{F}(f(t))] = f'(t) \vec{F}'(f(t))$

chain rule