

§ 12.6 Cylinders and Quadric Surfaces

12.6.1

Review from 12.1

• Examples of Surfaces we have seen so far:

1. sphere < in 12.1 >

2. cylinder < in 12.1 >

3. plane < in 12.1 > < then more in 12.5 >

→ Review Desmos Demos (DD):

• 12.6.1 Examples of Cylinders

• 12.6.2 Cylinders and Traces

Def The trace on a surface $\mathbb{S}$ by another surface $\mathbb{P}$ (often a plane) is the set of points on $\mathbb{S}$ which are also on $\mathbb{P}$.

Note 1. trace on $\mathbb{S}$ by $\mathbb{P} = \mathbb{S} \cap \mathbb{P}$

2. trace "sits" on $\mathbb{S}$ in $\mathbb{R}^3$.

3. A trace by a plane is also called a cross-section.

Ex Fill-in blank.

The trace on the cylinder $x^2 + y^2 = 4$ by the plane $z = my$ is:
an ellipse.

Soln. Review DD 12.1.4 Paper Towel roll.

§ 12.6

Def A quadric surface is a surface defined by a 2nd - degree equation in x, y , and z .

so is a sum of terms of the form: $c x^p y^q z^r$
where $c \in \mathbb{R}$ and $p, q, r \in \{0, 1, 2\}$ and $p + q + r \leq 2$

Exs $x^2 + y^2 + z^2 = 17$ and $3x + 4y + 5z = 6$ and $4xy + 5z = 8$.

Ex 1 Let the surface $\mathcal{E}$ be given by

$$\frac{x^2}{2^2} + \frac{y^2}{3^2} + \frac{z^2}{4^2} = 1.$$

1.1 Describe the trace on $\mathcal{E}$ by the plane $z=0$. (i.e. by xy -plane)

Soln $\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1$ and $z=0$.

$\Rightarrow$ an ellipse in the $z=0$ plane that goes thru the (key) points
(when $y=0$) $(\pm 2, 0, 0)$ and (when $x=0$) $(0, \pm 3, 0)$.

1.2 Describe the trace on $\mathcal{E}$ by the plane $z=k$. (so plane is $\parallel$ to xy -plane)

$$\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1 - \frac{k^2}{4^2} \quad \text{and} \quad z=k.$$

Case $1 - \frac{k^2}{4^2} = 0 \quad \left(\begin{array}{l} \text{note} \\ \Leftrightarrow k = \pm 4 \end{array} \right)$

$$\Rightarrow \frac{x^2}{2^2} + \frac{y^2}{3^2} = 0 \Rightarrow x=0=y.$$

Trace is the pt
 $(0, 0, k)$

so get $(0, 0, +4)$
and $(0, 0, -4)$.

Case $1 - \frac{k^2}{4^2} < 0 \quad (\Leftrightarrow |k| > 4)$

$$\Rightarrow \frac{x^2}{2^2} + \frac{y^2}{3^2} = 1 - \frac{k^2}{4^2}$$

Trace is the
empty set $\emptyset$.

Case $1 - \frac{k^2}{4^2} > 0 \quad (\Leftrightarrow |k| < 4) \Rightarrow$ The trace is the
ellipse $\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1 - \frac{k^2}{4^2}$ in the plane $z=k$.

1.3 For you: do the same as in 1.2 but for $x=k$ and $y=k$.

All traces will be either: an ellipse, a point, or $\emptyset$.

For help, see DD 12.6.4 (Ellipsoid via a few traces by plane $\parallel$ to coord. planes).

1.4 Do DD 12.6.3 (Ellipsoid via a lot of traces by planes $z = \text{a constant}$).

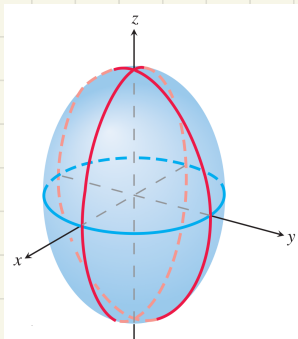
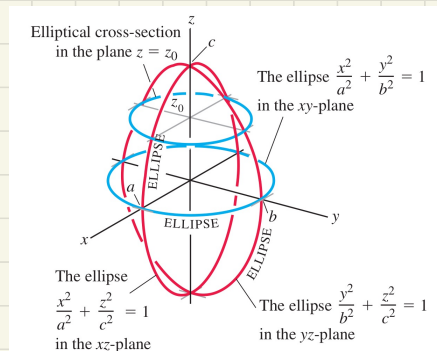
1.5 HW: Do DD 12.6.5 (Ellipsoid via lots of traces by planes $\parallel$ to coord planes)

ELLIPSOID

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

from our book

§ 12.6



Hyperbolic Paraboloid (saddle)

12.6.3

for constants a, b , and c with $c > 0$

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}$$

← Note, c not c^2 .

$$\text{So } z = \frac{c}{b^2} y^2 - \frac{c}{a^2} x^2.$$

1. Trace by the plane $x=0$ is $z = \frac{c}{b^2} y^2$ } parabola
 Trace by the plane $x=x_0$ is $z = \frac{c}{b^2} y^2 - \underbrace{\frac{cx_0^2}{a^2}}_{\text{a constant}}$ } CCU

2. Trace by the plane $y=0$ is $z = -\frac{c}{a^2} x^2$ } parabola
 Trace by the plane $y=y_0$ is $z = -\frac{c}{a^2} x^2 + \underbrace{\frac{cy_0^2}{b^2}}_{\text{a constant}}$ } CCD

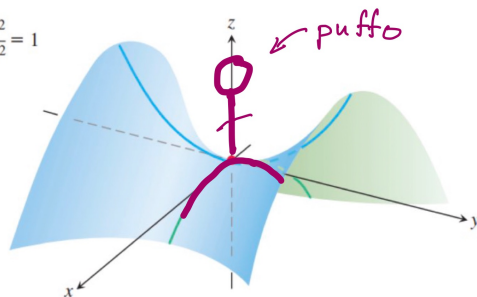
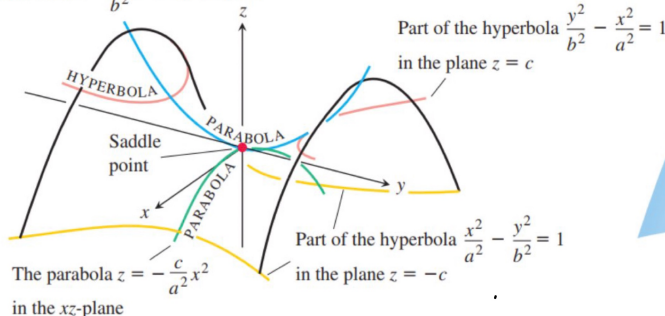
3. Trace by the plane $z=0$ is $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 0 \Rightarrow y^2 = \frac{b^2}{a^2} x^2 \Rightarrow y = \pm \frac{b}{a} x$
 So trace by $z=0$ is the 2 lines $y = \pm \frac{b}{a} x$ thru $(0,0)$.

Trace by the plane $z=z_0$ when $z_0 \neq 0$ is (do the algebra)

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z_0}{c} \Rightarrow \frac{c}{z_0 b^2} y^2 - \frac{c}{z_0 a^2} x^2 = 1 \leftarrow \text{hyperbola.}$$

- The saddle is symmetric abt the planes $x=0$ and $y=0$. (no change if replace x w/ $-x$).
 A (color-coded) sketch, from our book,

The parabola $z = \frac{c}{b^2} y^2$ in the yz -plane



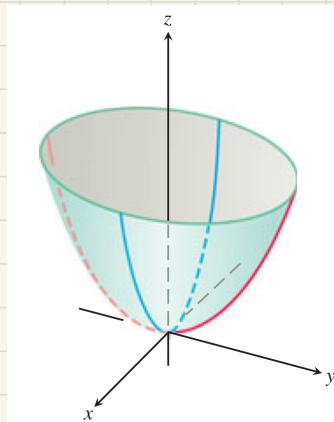
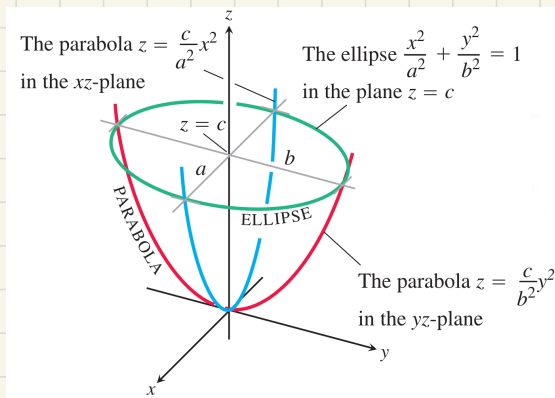
The saddle point is $(0,0,0)$.

- * If a puffo is sitting on the saddle $\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}$ then:
- his legs are hanging over the CCD parabola $z = -\frac{c}{a^2} x^2$ in the xy -plane
 - rest of his body is on positive z -axis.
- Do Desmos Demo 12.6 on Hyperbolic Paraboloid (saddle).

- 12.6.4
- Two other quadric surfaces we will use for class examples.
Be able to make rough sketches of by looking at their traces.
Figures are from book (12.6)

PARABOLOID $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$

$c > 0$



CONE $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$

