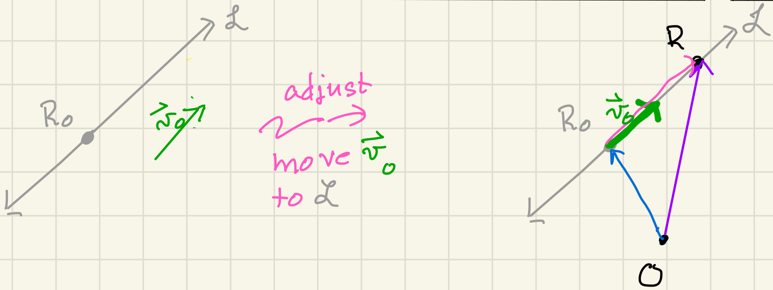
12.5 Lines and Planes (in 3D)

Line in 3D

A line $\mathcal{L}$
- through the point $R_0 = (x_0, y_0, z_0)$

and
- parallel to $\vec{v}_0 = \langle a, b, c \rangle$, with $\vec{v}_0 \neq \vec{0}$



Let $R = (x, y, z)$ be a point on $\mathcal{L}$. Put in the origin O. Note

$$\overrightarrow{OR} = \overrightarrow{OR_0} + t\,\vec{v}_0 \quad , \text{where } -\infty < t < \infty$$

Think of t as time. When $t = 0$, we are at R_0.
What happens as $t \nearrow \infty$. What happens as $t \searrow -\infty$.

Vector Equation of $\mathcal{L}$:

$$\boxed{\vec{R}(t) = \vec{R}_0(t) + t\,\vec{v}_0 \quad , \quad -\infty < t < \infty.} \qquad (1)$$

In long form, (1) says

$$\langle x(t), y(t), z(t) \rangle = \langle x_0, y_0, z_0 \rangle + t\,\langle a, b, c \rangle \qquad (2)$$

In (2) adding the upper side and equating coordinates, we get

Parametric Equation of $\mathcal{L}$

$$\boxed{x(t) = x_0 + at \quad \text{and} \quad y(t) = y_0 + bt \quad \text{and} \quad z(t) = z_0 + ct, \quad -\infty < t < \infty} \qquad (3)$$

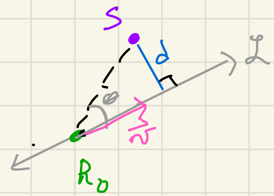
If $abc \neq 0$, then (3) is equiv. to $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$.

Can think of (1) as

$$\vec{R}(t) = \underset{\text{initial position}}{\vec{R}_0(t)} + \underset{\text{time}}{t}\,\underset{\text{speed}}{\|\vec{v}_0\|}\,\underbrace{\frac{\vec{v}_0}{\|\vec{v}_0\|}}_{\text{direction (a unit vector)}}$$

• The distance d from a point S to the line $\mathcal{L}$ thru point R_0 and parallel to $\vec{v}_0$ is 12.5.2.

$$\boxed{d = \left\| \overrightarrow{R_0S} \times \frac{\vec{v}_0}{\|\vec{v}_0\|} \right\|}$$



since

$$\left\| \overrightarrow{R_0S} \times \frac{\vec{v}_0}{\|\vec{v}_0\|} \right\| = \|\overrightarrow{R_0S}\| \left\| \frac{\vec{v}_0}{\|\vec{v}_0\|} \right\| \left|\sin\theta_{\overrightarrow{R_0S},\vec{v}}\right| \; \|\vec{n}\|$$

$$= \|\overrightarrow{R_0S}\| \left|\sin\theta_{\overrightarrow{R_0S},\vec{v}}\right| = \|\overrightarrow{R_0S}\| \frac{d}{\|\overrightarrow{R_0S}\|} = d.$$

Plane in 3D

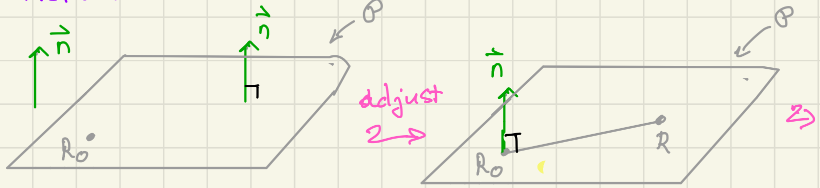
(PC) = plug and chug — 12.5.3

- The plane $\mathcal{P}$
  - containing the point $R_0 = (x_0, y_0, z_0)$

  and

  - is normal ($\perp$) to the vector $\vec{n} = \langle a, b, c \rangle$, with $\vec{v} \neq \vec{0}$

Picture



Note

$R = (x, y, z) \in \mathcal{P}$ if and only if $\vec{n} \cdot \overrightarrow{R_0R} = 0$

This plane $\mathcal{P}$ has

- vector equation $\quad \vec{n} \cdot \overrightarrow{R_0R} = 0$ $\qquad \boxed{\overrightarrow{R_0R} = \langle x - x_0, y - y_0, z - z_0 \rangle}$

(PC) ↓

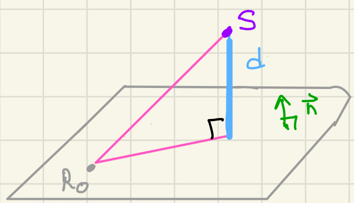
- Component equation $\quad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

(PC) ↓

- Comp. eq. simplified $\quad ax + by + cz = d \quad$ where $\quad d = ax_0 + by_0 + cz_0$

- The distance d from a point S to the plane thru the pt. R_0 and with normal $\vec{n}$

Picture



$$d = \left| \operatorname{proj}_{\vec{n}} \overrightarrow{R_0S} \right|$$

$$= \left| \operatorname{comp}_{\vec{n}} \overrightarrow{R_0S} \right|$$

$$= \overrightarrow{R_0S} \cdot \frac{\vec{n}}{\|\vec{n}\|}$$

is

$$\boxed{d = \left| \overrightarrow{R_0S} \cdot \frac{\vec{n}}{\|\vec{n}\|} \right|}$$

Intersection of 2 planes

- **Set-up** Let $\mathcal{P}_1$ have normal vector $\vec{n}_1$.
  Let $\mathcal{P}_2$ have normal vector $\vec{n}_2$.

- **Question** What is the intersection of $\mathcal{P}_1$ and $\mathcal{P}_2$?
  ↳ denoted $\mathcal{P}_1 \cap \mathcal{P}_2$

1. If $\mathcal{P}_1 = \mathcal{P}_2$, then $\mathcal{P}_1 \cap \mathcal{P}_2 = \mathcal{P}_1$

2. If $\mathcal{P}_1 \neq \mathcal{P}_2$ but $\mathcal{P}_1 \parallel \mathcal{P}_2$, then $\mathcal{P}_1 \cap \mathcal{P}_2 = \emptyset \stackrel{\text{i.e.}}{=}$ empty set,

3. The interesting case. $\boxed{\text{Let } \mathcal{P}_1 \neq \mathcal{P}_2 \text{ and } \mathcal{P}_1 \nparallel \mathcal{P}_2}$.
   $\Updownarrow$
   $\dfrac{\vec{n}_1}{\|\vec{n}_1\|} \neq \pm \dfrac{\vec{n}_2}{\|\vec{n}_2\|}$

   - pipe cleansers help!

Then $\mathcal{P}_1 \cap \mathcal{P}_2$ is a line $\mathcal{L}$ with $\mathcal{L} \parallel (\vec{n}_1 \times \vec{n}_2)$

Def The angle between $\mathcal{P}_1$ and $\mathcal{P}_2$ is the $\underbrace{\text{acute}}_{\text{so btw. } 0 \text{ and } \pi/2.}$ angle between $\vec{n}_1$ and $\vec{n}_2$.