§12.4 Cross Product

$\boxed{\text{Set-up}}$ • Given 2 (nonzero) vectors

$$\vec{A} = \langle a_1, a_2, a_3 \rangle \quad \text{and} \quad \vec{B} = \langle b_1, b_2, b_3 \rangle .$$

• If needed, translate one of the vectors so that their tails coincide.

• Let

θ_{AB} be the angle between $\vec{A}$ and $\vec{B}$.

so

$$0 \leq \theta_{AB} \leq \pi$$

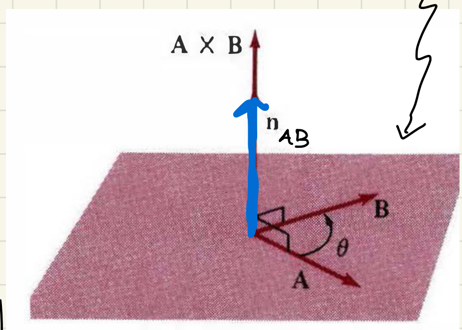
• Note $\vec{A} \parallel \vec{B} \iff \theta_{AB}$ is 0 are π

$\vec{A} \nparallel \vec{B} \iff 0 < \theta_{A} < \pi$.

• Now assume $\vec{A} \nparallel \vec{B}$; so, get $\vec{A}$ & $\vec{B}$ are contained in a unique plane $\mathcal{P}_{AB}$.

• Def

$\vec{n}_{AB}$ is the right-hand-rule unit vector perpendicular ($\perp$) to the plane $\mathcal{P}_{AB}$.



• Since $\vec{n}_{AB}$ is a unit vector, $\boxed{\|\vec{n}_{AB}\| = 1}$.

• Since $\vec{n}_{AB} \perp \mathcal{P}_{AB}$, we get $\boxed{\vec{n}_{AB} \perp \vec{A}}$ and $\boxed{\vec{n}_{AB} \perp \vec{B}}$

• Right-hand-rule stuff for $\vec{n}_{AB}$:

Place your right hand $\perp$ to $\mathcal{P}_{AB}$ and along $\vec{A}$.

Curl your fingers from $\vec{A}$ to $\vec{B}$, going through θ_{AB}.

Your thumb will be pointing in the direction of $\vec{n}_{AB}$.

Question: $\vec{n}_{AB}$ vs. $\vec{n}_{BA}$?

$\boxed{\text{Def.}}$ Cross Product

$$\underbrace{\vec{A} \times \vec{B}}_{\text{vectors}} = \underbrace{\left[\|\vec{A}\| \, \|\vec{B}\| \sin \theta_{AB} \right] \vec{n}_{AB}}_{\text{a vector}}$$

Recall $\boxed{\text{Def.}}$ Cross Product

$$\underset{\text{vectors}}{\vec{A} \times \vec{B}} = \underbrace{\left[\|\vec{A}\| \, \|\vec{B}\| \sin\theta_{AB}\right] \vec{n}_{AB}}_{\text{a vector}} \qquad (1)$$

- Using the definition in (1), we get the following:

Algebraic Properties of the Cross Product

5. $\vec{0} \times \vec{B} = \vec{0}$
3. $\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A})$
1. $(r\vec{A}) \times (s\vec{B}) = (rs)(\vec{A} \times \vec{B})$
2. $\vec{A} \times (\vec{B} + \vec{C}) = (\vec{A} \times \vec{B}) + (\vec{A} \times \vec{C})$
4. $(\vec{A} + \vec{B}) \times \vec{C} = (\vec{A} \times \vec{C}) + (\vec{B} \times \vec{C})$
6. $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$

- Using these algebraic properties, we can show (see book p 736) for $\vec{A} = \langle x_A, y_A, z_A \rangle$ and $\vec{B} = \langle x_B, y_B, z_B \rangle$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_A & y_A & z_A \\ x_B & y_B & z_B \end{vmatrix} \quad \text{i.e. det.} \left(\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_A & y_A & z_A \\ x_B & y_B & z_B \end{bmatrix} \right)$$

$$\text{i.e. } (y_A z_B - z_A y_B)\vec{i} - (x_A z_B - z_A x_B)\vec{j} + (x_A y_B - y_A x_B)\vec{k}$$

do not overlook minus sign.

Do not have to memorize! Instead, see Handout on: Determinate of a Matrix.

Ex 1. Let $\vec{A} = 2\vec{i} - \vec{j} + 4\vec{k}$ and $\vec{B} = \vec{i} + 5\vec{j} - 3\vec{k}$. Fill in the blanks

(1a) $\vec{A} \times \vec{B} = \_\_\_\_\_\_ \vec{i} + \_\_\_\_\_\_ \vec{j} + \_\_\_\_\_\_ \vec{k}$

(1b) the direction of $\vec{A} \times \vec{B} = \langle \_\_\_\_\_\_ , \_\_\_\_\_\_ , \_\_\_\_\_\_ \rangle$

remember,
direction of $\vec{v}$ means unit vector in the same direction as $\vec{v}$ so want $\dfrac{\vec{v}}{\|\vec{v}\|}$.

Soln

(1a) $\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 4 \\ 1 & 5 & -3 \end{vmatrix}$

Recall (*) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

think

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 4 \\ 1 & 5 & -3 \end{vmatrix} \quad \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 4 \\ 1 & 5 & -3 \end{vmatrix} \quad \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 4 \\ 1 & 5 & -3 \end{vmatrix}$$

$= \vec{i} \begin{vmatrix} -1 & 4 \\ 5 & -3 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & 4 \\ 1 & -3 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & -1 \\ 1 & 5 \end{vmatrix}$

(*)

$= \vec{i}\left((-1)(-3) - (4)(5)\right) - \vec{j}\left((2)(-3) - (4)(1)\right) + \vec{k}\left((2)(5) - (-1)(1)\right)$

$= \vec{i}\,(3 - 20) - \vec{j}\,(-6 - 4) + \vec{k}\,(10 + 1)$

$= -17\,\vec{i} + 10\,\vec{j} + 11\,\vec{k}$

(1b) Want $\dfrac{\vec{A} \times \vec{B}}{\|\vec{A} \times \vec{B}\|}$. $\|\vec{A} \times \vec{B}\| = \sqrt{17^2 + 10^2 + 11^2} = \sqrt{289 + 100 + 121}$

$= \sqrt{510}$

$\dfrac{\vec{A} \times \vec{B}}{\|\vec{A} \times \vec{B}\|}$ i.e. $\dfrac{1}{\|\vec{A} \times \vec{B}\|}(\vec{A} \times \vec{B}) = \dfrac{1}{\sqrt{510}}\langle -17, 10, 11 \rangle = \left\langle \dfrac{-17}{\sqrt{510}}, \dfrac{10}{\sqrt{510}}, \dfrac{11}{\sqrt{510}} \right\rangle$

Ex 2 Neat Useful Fact for nonzero vectors $\vec{A}$ and $\vec{B}$.

1. $\vec{A} \cdot \vec{B} = 0 \iff$ ____________

2. $\vec{A} \times \vec{B} = \vec{0} \iff$ ____________

Soln

1. $\vec{A} \cdot \vec{B} = 0 \iff \|\vec{A}\| \, \|\vec{B}\| \cos\theta_{AB} = 0 \iff \cos\theta_{AB} = 0$

but $0 \le \theta_{AB} \le \pi$

$\iff \theta = \pi/2 \iff A \perp B$

2. $\vec{A} \times \vec{B} = \vec{0} \iff \|\vec{A}\| \, \|\vec{B}\| \sin\theta_{AB} \, \vec{n}_{AB} = \vec{0}$

$\iff \big\| \, \|\vec{A}\| \, \|\vec{B}\| \sin\theta_{AB} \, \vec{n}_{AB} \, \big\| = \|\vec{0}\|$

$\iff \|\vec{A}\| \, \|\vec{B}\| \, |\sin\theta_{AB}| \, \cancelto{1}{\|\vec{n}_{AB}\|} = \|\vec{0}\|$

$\iff |\sin\theta_{AB}| = 0$ (know $0 \le \theta_{AB} \le \pi$)

$\iff \theta_{AB}$ is 0 or π. $\iff \vec{A} \parallel \vec{B}$

Ex 3 Application. Torque is given by a cross product.

and engineering

The cross product arises quite naturally in many situations in physics. For example, if a force **F** is applied to a body at a point P (Fig. 18.19), and if **R** is the vector from a fixed origin O to P, then this force tends to rotate the body about an axis through O and perpendicular to the plane of **R** and **F**. The *torque vector* **T** defined by

$$\mathbf{T} = \mathbf{R} \times \mathbf{F}$$

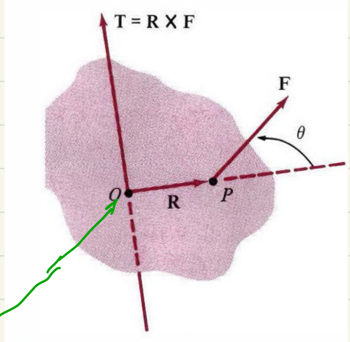


Figure 18.19 Torque vector.

Think of the body as being fixed at O but the body can still rotate about O.