§ 12.3 Dot Product

Defs / Set-up

- Given vectors $\vec{A}$ and $\vec{B}$

in 3D

$\vec{A} = \langle x_A, y_A, z_A \rangle$

$\vec{B} = \langle x_B, y_B, z_B \rangle$

in 2D

$\vec{A} = \langle x_A, y_A \rangle$

$\vec{B} = \langle x_B, y_B \rangle$

To put 2D into 3D:

$\xrightarrow{\text{think}} \langle x_A, y_A, 0 \rangle$

$\xrightarrow{\text{think}} \langle x_B, y_B, 0 \rangle$

Similarly for a vector $\vec{C}$.

- Def For nonzero $\vec{A}$ and $\vec{B}$

θ_{AB} = the (smallest) angle between $\vec{A}$ and $\vec{B}$

Note $0 \le \theta_{AB} \le \pi$

- Def The dot product of $\vec{A}$ and $\vec{B}$ is

$$\boxed{\vec{A} \bullet \vec{B} = x_A x_B + y_A y_B + z_A z_B} \qquad (1)$$

- Fact For nonzero $\vec{A}$ and $\vec{B}$ (see book p. 727)

$$\boxed{\vec{A} \bullet \vec{B} = |\vec{A}|\,|\vec{B}| \cos\theta_{AB}} \qquad (2)$$

so $\boxed{\cos\theta_{AB} = \dfrac{\vec{A} \bullet \vec{B}}{|\vec{A}|\,|\vec{B}|}}$ and $\boxed{\theta_{AB} = \cos^{-1}\left(\dfrac{\vec{A} \cdot \vec{B}}{|\vec{A}|\,|\vec{B}|}\right)}$

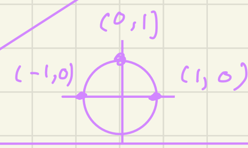
recall $0 \le \cos^{-1}(\text{any angle}) \le \pi$

⚠ The dot product of 2 vectors is a number. (i.e. scalar)

- Def $\vec{A}$ and $\vec{B}$ are orthogonal ($\perp$) $\iff$ $\vec{A} \bullet \vec{B} = 0$

Note

- if $\vec{A} = \vec{0}$ then $\vec{A} \cdot \vec{B} \xlongequal{\text{look at (1)}} 0$.
- if $\vec{B} = \vec{0}$ then $\vec{A} \cdot \vec{B} \xlongequal{\text{look at (1)}} 0$.
- if $\vec{A}$ and $\vec{B}$ are nonzero then



$\vec{A} \cdot \vec{B} = 0 \xLeftrightarrow{\text{look at (2)}} |\vec{A}|\,|\vec{B}| \cos\theta_{AB} = 0 \iff \cos\theta_{AB} = 0 \iff \theta_{AB} = \frac{\pi}{2} \iff \vec{A} \perp \vec{B}$

- Recall $\boxed{\vec{A}\cdot\vec{B} = x_A x_B + y_A y_B + z_A z_B}$

So $\boxed{\vec{A}\cdot\vec{A} = |\vec{A}|^2}$

- <u>Algebraic Properties of dot Product</u> < think/remember "do coord-wise">

For vectors $\vec{A}, \vec{B}, \vec{c}$ and a scalar k < so $k \in \mathbb{R}$>

1. $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$
2. $(k\vec{A})\cdot B = k(\vec{A}\cdot\vec{B}) = \vec{A}\cdot(k\vec{B})$
3. $\vec{A}\cdot(\vec{B}+\vec{C}) = (\vec{A}\cdot\vec{B}) + (\vec{A}\cdot\vec{C})$
4. $\vec{0}\cdot\vec{A} = 0$. Question: Is it correct to write $0\cdot\vec{A} = 0$? ____

- <u>Def / Application.</u> The <u>work</u> done by a constant force $\vec{F}$ acting along a displacement of $\vec{D}$ is $\boxed{W = \vec{F}\cdot\vec{D}}$

- To firm down "same direction"

<u>Def</u> The (three) <u>directional angles</u> of $\vec{A}$ are:

α = angle btw. $\vec{A}$ and $\vec{i}$, i.e. the positive x-axis
β = angle btw. $\vec{A}$ and $\vec{j}$ y-axis
γ = angle btw. $\vec{A}$ and $\vec{k}$ z-axis

- Think of the 3 directional angles as telling us how much $\vec{A}$ "swings" from the 3 (positive) coordinate axes.

- Note $0 \le \alpha, \beta, \gamma \le \pi$ and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

- <u>Also</u> Two vectors have the same direction $\Longleftrightarrow$ their corresponding directional angles are the same.

- For a nice drawing, see book p 732 Exercise 12.3.15.

- Calculation 9

$$\vec{A}\cdot\frac{\vec{B}}{|\vec{B}|} = \frac{1}{|\vec{B}|}(\vec{A}\cdot\vec{B}) = \frac{1}{|\vec{B}|}\left(|\vec{A}|\,|\vec{B}|\cos\theta_{AB}\right) = |\vec{A}|\cos\theta_{AB}$$

- Defs Let $\vec{A}$ and $\vec{B}$ be nonzero. <see book p 729>

- The vector projection of $\vec{A}$ onto $\vec{B}$ is (the vector)

$$\boxed{\begin{aligned}\operatorname{proj}_{\vec{B}}\vec{A} &= \left(\vec{A}\cdot\frac{\vec{B}}{|\vec{B}|}\right)\frac{\vec{B}}{|\vec{B}|}\\ &= \underbrace{\left(|\vec{A}|\cos\theta_{AB}\right)}_{\text{a \#}}\underbrace{\frac{\vec{B}}{|\vec{B}|}}_{\substack{\text{a vector}\\ \text{even a unit vector}}}\end{aligned}}$$

use Calculation 9

Recall $\frac{\vec{B}}{|\vec{B}|}$ is the unit vector in the direction of $\vec{B}$

(0,1) (−1,0) (1,0)

- let's look at pictures below.

- The (signed) component of $\vec{A}$ in the direction of $\vec{B}$ is (the #)

+/−

$$\boxed{\operatorname{comp}_{\vec{B}}\vec{A} = \left(\vec{A}\cdot\frac{\vec{B}}{|\vec{B}|}\right) = |\vec{A}|\cos\theta_{AB}}$$

use calculation 9

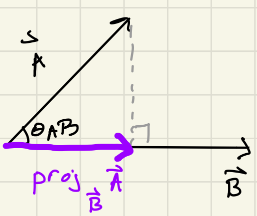
is $\begin{cases} >0, & \text{if } 0\le\theta_{AB}<\pi/2\\ =0, & \text{if } \theta_{AB}=\pi/2\\ <0, & \text{if } \frac{\pi}{2}<\theta_{AB}\le\pi.\end{cases}$

(*) Note the length of the vector $\operatorname{proj}_{\vec{B}}\vec{A}$ = the abs. value of $\operatorname{comp}_{\vec{B}}\vec{A}$

bcs $|\operatorname{proj}_{\vec{B}}\vec{A}| = \left|\vec{A}\cdot\frac{\vec{B}}{|\vec{B}|}\right| = |\vec{A}|\,|\cos\theta_{AB}| = |\operatorname{comp}_{\vec{B}}\vec{A}|$

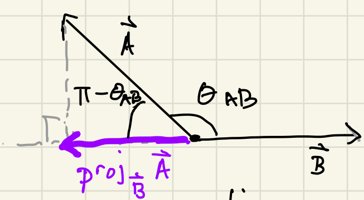
- Pictures

when $0<\theta_{AB}<\pi/2$



$$\cos\theta_{AB} = \frac{\text{adj}}{\text{hyp}} = \frac{|\operatorname{proj}_{\vec{B}}\vec{A}|}{|\vec{A}|}$$

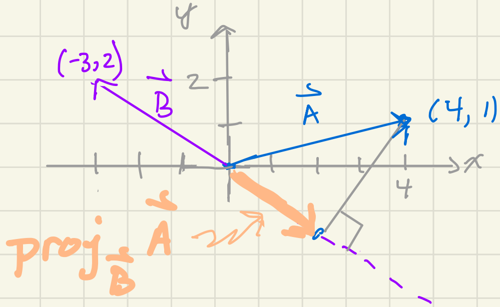
when $\pi/2<\theta_{AB}<\pi$



$$\underbrace{\cos\theta_{AB}}_{\text{negative}} = \underbrace{-\cos(\pi-\theta_{AB})}_{\text{positive}} = \frac{\text{adj}}{\text{hyp}} = \frac{|\operatorname{proj}_{\vec{B}}\vec{A}|}{|\vec{A}|}$$

12.3.4

Ex. Let $\vec{A} = 4\vec{i} + \vec{j} \overset{\text{i.e.}}{=} \langle 4, 1 \rangle$ and $\vec{B} = -3\vec{i} + 2\vec{j} \overset{\text{ie}}{=} \langle -3, 2 \rangle$



1. $|\vec{A}| =$ ____________________

2. $|\vec{B}| =$ ____________________

3. $\vec{A} \cdot \vec{B} =$ ____________________

4. $\cos \theta_{AB} =$ ____________________

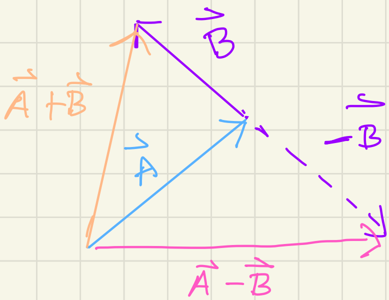
↳ Work:

5. $\text{proj}_{\vec{B}} \vec{A} =$ ____________________

6. $\text{comp}_{\vec{B}} \vec{A} =$ ____________________

Ex. If $\vec{A}$ and $\vec{B}$ are vectors (in $\mathbb{R}^2$ or $\mathbb{R}^3$), when are $(\vec{A}+\vec{B})$ and $(\vec{A}-\vec{B})$ orthogonal? Give necessary and suffient conditions on $\vec{A}$ and $\vec{B}$.

Solution $\boxed{}$



Recall $(\vec{A}+\vec{B})$ orthog. to $(\vec{A}-\vec{B})$

$\Leftrightarrow (\vec{A}+\vec{B}) \perp (\vec{A}-\vec{B})$

$\Leftrightarrow \cos(\measuredangle \text{ btw}) = 0.$

$\Leftrightarrow$ ____________________