We cannot take the product of 2 vectors. We can take their Dot Product as well as Cross Product.

►. Given vectors

$$\vec{A} = \langle x_A, y_A, z_A \rangle \qquad \text{and} \qquad \vec{B} = \langle x_B, y_B, z_B \rangle .$$

Then $\vec{A}$ dot product $\vec{B}$ is the scalar

$$\vec{A} \cdot \vec{B} \overset{\text{def}}{=} x_A x_B + y_A y_B + z_A z_B \tag{DP}$$

and $\vec{A}$ cross product $\vec{B}$ is the vector

$$\vec{A} \times \vec{B} \overset{\text{def}}{=} \det \begin{bmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ x_A & y_A & z_A \\ x_B & y_B & z_B \end{bmatrix} . \tag{CP}$$

Dot Product

1. Note $\vec{A} \cdot \vec{A} = (x_A)^2 + (y_A)^2 + (z_A)^2$ so

$$\vec{A} \cdot \vec{A} = \left\| \vec{A} \right\|^2 . \tag{1}$$

2. For nonzero vectors $\vec{A}$ and $\vec{B}$, let ⟨draw $\vec{A}$ and $\vec{B}$ tail-to-tail⟩

θ_{AB} be the (smallest) angle between $\vec{A}$ and $\vec{B}$. (so $0 \le \theta_{AB} \le \pi$ and $\theta_{AB} = \theta_{BA}$)

and then we get ⟨use the law of cosine, see book⟩

$$\vec{A} \cdot \vec{B} \overset{\text{def}}{=} x_A x_B + y_A y_B + z_A z_B = \left\| \vec{A} \right\| \left\| \vec{B} \right\| \cos \theta_{AB} \tag{2}$$

and so

$$\theta_{AB} = \arccos \left(\frac{\vec{A} \cdot \vec{B}}{\left\| \vec{A} \right\| \left\| \vec{B} \right\|} \right) . \tag{3}$$

3. Properties of Dot Product. Given vectors: $\vec{A}$, $\vec{B}$, $\vec{C}$. Given scalars: r, s.

(1) $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$

(2) $\vec{0} \cdot \vec{B} = 0$.

(3) $\left(r\vec{A} \right) \cdot \left(s\vec{B} \right) = (rs) \left(\vec{A} \cdot \vec{B} \right)$

(4) $\vec{A} \cdot \left(\vec{B} + \vec{C} \right) = \left(\vec{A} \cdot \vec{B} \right) + \left(\vec{A} \cdot \vec{C} \right)$

4. **Def**. Vectors $\vec{A}$ and $\vec{B}$ are orthogonal, denoted $\vec{A} \perp \vec{B}$, provided $\vec{A} \cdot \vec{B} = 0$.
By (2), $\vec{A} \perp \vec{B}$ precisely when at least one of the following hold: $\theta_{AB} = \frac{\pi}{2}$, $\vec{A} = \vec{0}$, $\vec{B} = \vec{0}$.

5. Projection (a vector) and Component (a scalar).
The projection of $\vec{A}$ onto $\vec{B}$ and the (signed) component of $\vec{A}$ in the direction of $\vec{B}$ are:

$$\overrightarrow{\text{proj}}_{\vec{B}}\, \vec{A} \overset{\text{def}}{=} \left(\vec{A} \cdot \frac{\vec{B}}{\left\| \vec{B} \right\|} \right) \frac{\vec{B}}{\left\| \vec{B} \right\|} \overset{(*)}{=} \left(\left\| \vec{A} \right\| \cos \theta_{AB} \right) \frac{\vec{B}}{\left\| \vec{B} \right\|} \tag{4}$$

$$\text{comp}_{\vec{B}}\, \vec{A} \overset{\text{def}}{=} \vec{A} \cdot \frac{\vec{B}}{\left\| \vec{B} \right\|} \overset{(*)}{=} \left\| \vec{A} \right\| \cos \theta_{AB} . \tag{5}$$

Note $\left\| \overrightarrow{\text{proj}}_{\vec{B}}\, \vec{A} \right\| = \left| \text{comp}_{\vec{B}}\, \vec{A} \right|$.

(∗) Calculation used: $\vec{A} \cdot \frac{\vec{B}}{\|\vec{B}\|} = \frac{1}{\|\vec{B}\|} \left(\vec{A} \cdot \vec{B} \right) = \frac{1}{\|\vec{B}\|} \left(\left\| \vec{A} \right\| \left\| \vec{B} \right\| \cos \theta_{AB} \right) = \left\| \vec{A} \right\| \cos \theta_{AB}$.

Cross Product

►. Recall definition of cross product:

$$\langle x_A, y_A, z_A \rangle \times \langle x_B, y_B, z_B \rangle \stackrel{\text{def}}{=} \det \begin{bmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ x_A & y_A & z_A \\ x_B & y_B & z_B \end{bmatrix}. \tag{CP}$$

Def. Two nonzero vectors are parallel if one is a nonzero scalar multiple of the other. ⟨beware skewed vectors⟩

6. Looking at the determinate in CP) we see:
The cross product of 2 parallel nonzero vectors is $\vec{0}$.
The cross product of 2 vectors, for which at least one them is the zero vector, is $\vec{0}$.

7. Put nonzero nonparallel vectors $\vec{A} = \langle x_A, y_A, z_A \rangle$ and $\vec{B} = \langle x_B, y_B, z_B \rangle$ with their tails at the origin. Then there is a unique plane $\mathcal{P}_{AB}$ thru the points: $(0,0,0)$ and (x_A, y_A, z_A) and (x_B, y_B, z_B). Then

$\vec{n}_{AB}$ is the right-hand-rule unit vector perpendicular to the plane $\mathcal{P}_{AB}$.

Note:

(1) $\vec{n}_{AB} \perp \vec{A}$
(2) $\vec{n}_{AB} \perp \vec{B}$
(3) $\|\vec{n}_{AB}\| = 1$
(4) $\vec{n}_{AB} = -\vec{n}_{BA}$

Also, as shown in book,

$$\vec{A} \times \vec{B} \stackrel{\text{def}}{=} \det \begin{bmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ x_A & y_A & z_A \\ x_B & y_B & z_B \end{bmatrix} = \left[\left\|\vec{A}\right\| \left\|\vec{B}\right\| \sin_{AB} \right] \vec{n}_{AB}. \tag{6}$$

8. Nonzero vectors $\vec{A}$ and $\vec{B}$ are parallel, denoted $\vec{A} \| \vec{B}$, if and only if $\vec{A} \times \vec{B} = \vec{0}$.

9. Properties of Cross Product. Given vectors: $\vec{A}$, $\vec{B}$, $\vec{C}$. Given scalars: r, s.

(1) $\vec{A} \times \vec{B} = -\left(\vec{B} \times \vec{A}\right)$

(2) $\vec{0} \times \vec{B} = \vec{0}$.

(3) $\left(r\vec{A}\right) \times \left(s\vec{B}\right) = (rs)\left(\vec{A} \times \vec{B}\right)$

(4) $\vec{A} \times \left(\vec{B} + \vec{C}\right) = \left(\vec{A} \times \vec{B}\right) + \left(\vec{A} \times \vec{C}\right)$

(5) $\left(\vec{A} + \vec{B}\right) \times \vec{C} = \left(\vec{A} \times \vec{C}\right) + \left(\vec{B} \times \vec{C}\right)$

(6) $\vec{A} \times \left(\vec{B} \times \vec{C}\right) \stackrel{(*)}{=} \left(\vec{A} \cdot \vec{C}\right)\vec{B} - \left(\vec{A} \cdot \vec{B}\right)\vec{C}$ (cross product is not associative)

(7) $\left(\vec{A} \times \vec{B}\right) \cdot \vec{C} \stackrel{(*)}{=} \vec{A} \cdot \left(\vec{B} \times \vec{C}\right)$ (triple scalar product)

A $\stackrel{(*)}{=}$ indicated that you do not need to memorize for exam but should be able to use if given.