Why the 2 formulas for Dot Product are equal.

Given nonzero vectors $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$.

Dot Product def.: $\vec{v} \cdot \vec{w} \overset{\text{def}}{=} v_1 w_1 + v_2 w_2 + v_3 w_3$. Let $\theta = \measuredangle$ btw. $\vec{v}$ and $\vec{w}$.

Now we will show

$$\boxed{\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\theta} \qquad (DP_2)$$

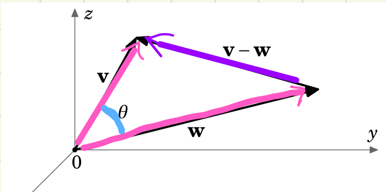
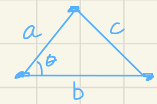


Figure 1

Law of Cosines



$c^2 = a^2 + b^2 - 2ab\cos\theta$

The calculation:

by Law of Cosines

$$\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\|\mathbf{v}\|\,\|\mathbf{w}\|\cos\theta = \|\mathbf{v} - \mathbf{w}\|^2$$

since $\mathbf{v} - \mathbf{w} = \langle v_1 - w_1, v_2 - w_2, v_3 - w_3 \rangle$

$$= (v_1 - w_1)^2 + (v_2 - w_2)^2 + (v_3 - w_3)^2$$
$$= (v_1^2 - 2v_1w_1 + w_1^2) + (v_2^2 - 2v_2w_2 + w_2^2) + (v_3^2 - 2v_3w_3 + w_3^2)$$
$$= (v_1^2 + v_2^2 + v_3^2) + (w_1^2 + w_2^2 + w_3^2) - 2(v_1w_1 + v_2w_2 + v_3w_3)$$

all algebra

$$= \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2(\mathbf{v} \cdot \mathbf{w})$$

So we just showed:

$$\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\|\mathbf{v}\|\,\|\mathbf{w}\|\cos\theta = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2(\mathbf{v} \cdot \mathbf{w})$$

all algebra again — just cancel.

So $$-2\|\mathbf{v}\|\,\|\mathbf{w}\|\cos\theta = -2(\mathbf{v} \cdot \mathbf{w})$$

So $$\|\mathbf{v}\|\,\|\mathbf{w}\|\cos\theta = \mathbf{v} \cdot \mathbf{w}$$

i.e. $$\boxed{\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\,\|\mathbf{w}\|\cos\theta} \qquad (DP_2)$$

Rmk Above covers the case that $\vec{v}$ and $\vec{w}$ are sides of a triangles. So we were assumed $\vec{v} \not\parallel \vec{w}$. What if $\vec{v} \parallel \vec{w}$? Then $\vec{w} = k\vec{v}$ for some $k \neq 0$.

So $\vec{v} \cdot \vec{w} = \vec{v} \cdot (k\vec{v}) = k(\vec{v} \cdot \vec{v}) = k\|\vec{v}\|^2 = \|\vec{v}\|^2 k$

and $\|\vec{v}\|\,\|\vec{w}\|\cos\theta = \|\vec{v}\|\,\|k\vec{v}\|\cos\theta = \|\vec{v}\|\,|k|\,\|\vec{v}\|\cos\theta = \|\vec{v}\|^2 |k| \cos\theta$

and $k = |k|\cos\theta$ since $[k > 0 \Rightarrow \theta = 0]$ and $[k < 0 \Rightarrow \theta = \pi]$. So (DP_2) holds.

Why the 2 formulas for Cross Product are equal.

Given nonzero nonparallel vectors $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$.

Cross product def.: $\vec{v} \times \vec{w} \overset{def}{=} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$. Let $\theta = \theta_{vw} = \measuredangle$ btw $\vec{v}$ and $\vec{w}$

Now we show that

$$\vec{v} \times \vec{w} = [\, \|\vec{v}\| \, \|\vec{w}\| \sin\theta \,] \, \vec{n}_{vw} \qquad (CP_2)$$

By def., $\vec{n}_{vw}$ is the unit vector in direction of $\vec{v} \times \vec{w}$.

The scalar $\|\vec{v}\| \, \|\vec{w}\| \sin\theta \geq 0$ since $0 \leq \theta \leq \pi$.

(*) So all we need to show is that length of $\vec{v} \times \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \sin\theta$

So all we need to show is that $\|\vec{v} \times \vec{w}\|^2 = \|\vec{v}\|^2 \|\vec{w}\|^2 \sin^2\theta$.

The calculation: Recall $\vec{v} \times \vec{w} = \langle v_2 w_3 - v_3 w_2, \, v_3 w_1 - v_1 w_3, \, v_1 w_2 - v_2 w_1 \rangle$.

$$\begin{aligned}
\|\mathbf{v} \times \mathbf{w}\|^2 &= (v_2w_3 - v_3w_2)^2 + (v_3w_1 - v_1w_3)^2 + (v_1w_2 - v_2w_1)^2 \\
&= v_2^2w_3^2 - 2v_2w_2v_3w_3 + v_3^2w_2^2 + v_3^2w_1^2 - 2v_1w_1v_3w_3 + v_1^2w_3^2 + v_1^2w_2^2 - 2v_1w_1v_2w_2 + v_2^2w_1^2 \\
&= v_1^2(w_2^2 + w_3^2) + v_2^2(w_1^2 + w_3^2) + v_3^2(w_1^2 + w_2^2) - 2(v_1w_1v_2w_2 + v_1w_1v_3w_3 + v_2w_2v_3w_3)
\end{aligned}$$

all thru here is algebra, with some clever steps.

and now adding and subtracting $v_1^2w_1^2$, $v_2^2w_2^2$, and $v_3^2w_3^2$ on the right side gives

$$\begin{aligned}
&= v_1^2(w_1^2 + w_2^2 + w_3^2) + v_2^2(w_1^2 + w_2^2 + w_3^2) + v_3^2(w_1^2 + w_2^2 + w_3^2) \\
&\quad - (v_1^2w_1^2 + v_2^2w_2^2 + v_3^2w_3^2 + 2(v_1w_1v_2w_2 + v_1w_1v_3w_3 + v_2w_2v_3w_3)) \\
&= (v_1^2 + v_2^2 + v_3^2)(w_1^2 + w_2^2 + w_3^2) \\
&\quad - ((v_1w_1)^2 + (v_2w_2)^2 + (v_3w_3)^2 + 2(v_1w_1)(v_2w_2) + 2(v_1w_1)(v_3w_3) + 2(v_2w_2)(v_3w_3))
\end{aligned}$$

so using $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$ for the subtracted term gives

$$\begin{aligned}
&= (v_1^2 + v_2^2 + v_3^2)(w_1^2 + w_2^2 + w_3^2) - (v_1w_1 + v_2w_2 + v_3w_3)^2 \\
&= \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 - (\mathbf{v} \cdot \mathbf{w})^2 \\
&= \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 \left(1 - \frac{(\mathbf{v} \cdot \mathbf{w})^2}{\|\mathbf{v}\|^2 \|\mathbf{w}\|^2}\right) \\
&= \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 (1 - \cos^2\theta) \\
&= \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 \sin^2\theta
\end{aligned}$$

factor out $\|v\|^2 \|w\|^2$

some calculus

alg. again.

$\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\theta$

θ is angle btw $\vec{v}$ and $\vec{w}$.

We just showed $\|\mathbf{v} \times \mathbf{w}\|^2 = \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 \sin^2\theta$. So all done!

[Source. Vector Calc. by Corral. p: 16, 21-22] 250101