1. A vector represents a quantity (eg, displacement, velocity, or force) that has both a

1. magnitude (ie, length)
2. direction.

2. A vector $\vec{v}$ sitting in $\mathbb{R}^n$ is drawn/respresented by a directed line segment which goes from an initial (or start/tail) point $A \in \mathbb{R}^n$ to a terminal (or end/head) point $B \in \mathbb{R}^n$. Write: $\vec{v} = \overrightarrow{AB}$

The magnitude (or length) of the vector $\vec{v}$ is the length of the line segment from A to B.

The direction of the vector $\vec{v}$ is the direction of the directed line segment from A to B.

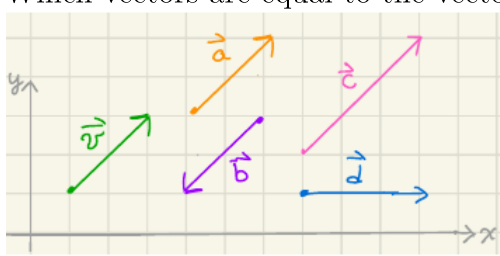
The vector above with the name of the letter "v" is written: $\vec{v}$ (arrow above) or $\mathbf{v}$ (boldface) or $\overrightarrow{AB}$.

⚠. $\vec{v} \neq v$!!! Do not forget your (needed) arrow above.

3. Two vectors are equal if and only if they have the same magnitude and direction.

⚠. So if vectors $\vec{v} = \overrightarrow{AB}$ and $\vec{u} = \overrightarrow{CD}$ have the same length and direction, then $\vec{v} = \vec{u}$ (even if the initial points of $\vec{v}$ and $\vec{u}$ are different and their terminal points are different).

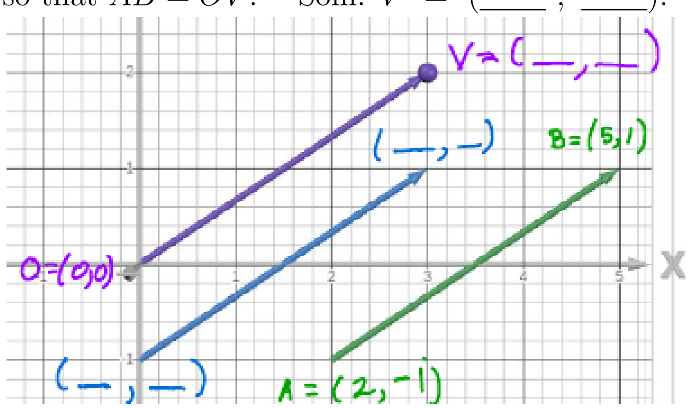
Ex1. (Working in the Cartesian plane $\mathbb{R}^2$.) Which vectors are equal to the vector $\vec{v}$? Soln: $\vec{v}$ = ________.



Def. Given a point $V = (x_V, y_V, z_V)$, the vector $\langle x_V, y_V, z_V \rangle \overset{\text{def}}{=} \overrightarrow{OV}$. ⚠Notice diff. btw: ( ) and ⟨ ⟩.
The component form of a vector is: $\langle x, y, x \rangle$ for $\mathbb{R}^3$ and $\langle x, y \rangle$ for $\mathbb{R}^2$.

Def. The standard position of $\overrightarrow{AB}$ is $\overrightarrow{OV}$ where O is the origin and $\overrightarrow{AB} = \overrightarrow{OV}$.

Ex2a. Given points $A = (2, -1)$ and $B = (5, 1)$ and $O = (0, 0)$ in $\mathbb{R}^2$.
Find the point $V \in \mathbb{R}^2$ so that $\overrightarrow{AB} = \overrightarrow{OV}$. Soln: V = (____ , ____).



Ex2b. Given points $A = (2, -1, -1)$ and $B = (5, 1, 2)$ and $O = (0, 0, 0)$ in $\mathbb{R}^3$.
Find the point $V \in \mathbb{R}^3$ so that $\overrightarrow{AB} = \overrightarrow{OV}$. Let's go to Desmos 12.2.1. Soln: V = (____ , ____ , ____).

4. Take aways from Desmos 12.2.1. When in $\mathbb{R}^2$ just do not write the z coord and geometrically think of as in $\mathbb{R}^3$ with $z = 0$.

∘. If $A = (x_A, y_A, z_A)$ and $B = (x_B, y_B, z_B)$ and the origin $O = (0,0,0)$ and V are points in $\mathbb{R}^3$, then the vectors $\overrightarrow{AB} = \overrightarrow{OV}$ when the point $V = (x_B - x_A, y_B - y_A, z_B - z_A)$, in which case $\overrightarrow{OV} \stackrel{\text{def}}{=} \langle x_B - x_A, y_B - y_A, z_B - z_A \rangle$ and so $\boxed{\overrightarrow{AB} = \langle x_B - x_A, y_B - y_A, z_B - z_A \rangle}$.

∘. For points the origin $O = (0,0,0)$ and

$$A = (x_A, y_A, z_A) \quad \text{and} \quad B = (x_B, y_B, z_B) \quad \text{and} \quad V = (x_V, y_V, z_V) \tag{1}$$

we get that $\boxed{\overrightarrow{AB} = \overrightarrow{OV}}$ if and only if

$$\begin{aligned} x_V &= x_B - x_A \\ y_V &= y_B - y_A \\ z_V &= z_B - z_A, \end{aligned} \tag{2}$$

in which case, since the length of a vector is the distance btw. its endpoints, the length of $\overrightarrow{AB}$, denoted $\left|\overrightarrow{AB}\right|$, is

$$\left|\overrightarrow{AB}\right| \stackrel{\text{def}}{=} d(A,B) \stackrel{\S 12.1}{=} \sqrt{|x_B - x_A|^2 + |y_B - y_A|^2 + |z_B - z_A|^2}$$

$$\stackrel{\text{by (2)}}{=} \sqrt{|x_V|^2 + |y_V|^2 + |z_V|^2} \underset{\S 12.1}{\stackrel{(1)}{=}} d(O,V) \stackrel{\text{def}}{=} \left|\overrightarrow{OV}\right| = |\langle x_V, y_V, z_V \rangle| .$$

5. Note: $|\vec{v}| = 0 \quad \Leftrightarrow \quad \vec{v} = \langle 0,0,0 \rangle$. Notation: $\langle 0,0,0 \rangle = \vec{0}$. Similarily in $\mathbb{R}^2$.

6. **Def**. Vector Addition, Scalar Multipication, and Vector Substraction.
Consider the vectors: $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ in $\mathbb{R}^3$.
Consider the scalar $k \in \mathbb{R}$. (Def. A scalar is a real number.)

- $\vec{u} + \vec{v} \stackrel{\text{def}}{=} \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$ (Vector Addition)
- $k\vec{u} \stackrel{\text{def}}{=} \langle ku_1, ku_2, ku_3 \rangle$ (Scalar Multipication)

And for Vector Subtraction

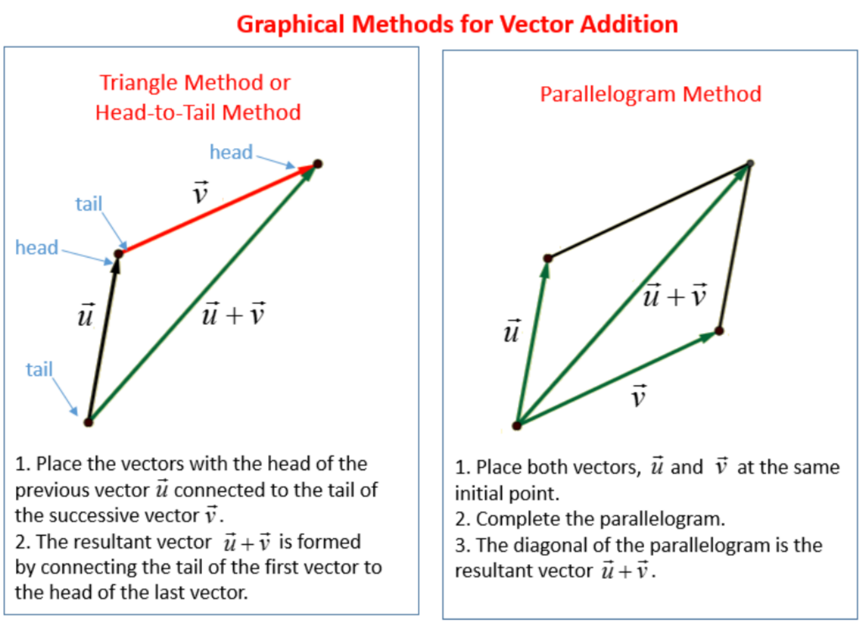
- $\vec{u} - \vec{v} \stackrel{\text{def}}{=} \vec{u} + ((-1)\,\vec{v}) \stackrel{\text{so}}{=} \langle u_1, u_2, u_3 \rangle + \langle -v_1, -v_2, -v_3 \rangle \stackrel{\text{so}}{=} \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle$

⚠. Note: can NOT multiple two vectors!

∘. When in $\mathbb{R}^2$, as above, just do not write the z coordinate and geometrically think of as in $\mathbb{R}^3$ with $z = 0$ (so in xy-plane).

∘. Algebraically: just *work component-wise.*

∘. Geometrically: to add two vectors, put the vectors *head-to-tail.*



Ex. Let $\vec{u} = \langle 4, 3 \rangle$ and $\vec{v} = \langle 1, -2 \rangle$. Find the following vectors. Express answer in component form.

- $\frac{1}{2}\vec{u} =$ ______________________
- $-\vec{v} =$ ______________________
- $\vec{u} + \vec{v} =$ ______________________
- $\vec{u} - \vec{v} =$ ______________________
- $\vec{u}\,\vec{v} =$ Silly Prof.! We can NOT multiply vectors !!

Ex. Let $\vec{u} = \langle 2, 2, 1 \rangle$ and $\vec{v} = \langle -2, -1, 1 \rangle$. Find the following vectors. Express answer component form.

- $\frac{1}{2}\vec{u} =$ ______________________
- $-\vec{v} =$ ______________________
- $\vec{u} + \vec{v} =$ ______________________
- $\vec{u} - \vec{v} =$ ______________________

7. Def. The vector $\vec{v}$ is a unit vector if and only if $|\vec{v}| = 1$.

8. Def. Standard unit (basis) vectors

$\mathbb{R}^3$. In $\mathbb{R}^3$, the standard unit vectors are:

$$\vec{\imath} \stackrel{\text{def}}{=} \langle 1,0,0\rangle \qquad \text{and} \qquad \vec{\jmath} \stackrel{\text{def}}{=} \langle 0,1,0\rangle \qquad \text{and} \qquad \vec{k} \stackrel{\text{def}}{=} \langle 0,0,1\rangle .$$

Any vector $\vec{v} = \langle v_1, v_2, v_3\rangle$ can be written as a linear combination of the standard unit vectors as follows:

$$\begin{aligned} \vec{v} &= \langle v_1, v_2, v_3\rangle \\ &= \langle v_1, 0, 0\rangle + \langle 0, v_2, 0\rangle + \langle 0,0,v_3\rangle \\ &= v_1\langle 1,0,0\rangle + v_2\langle 0,1,0\rangle + v_3\langle 0,0,1\rangle \\ &= v_1\,\vec{\imath} + v_2\,\vec{\jmath} + v_3\,\vec{k} \end{aligned}$$

with $v_1, v_2, v_3 \in \mathbb{R}$. The scalar v_1 is the i-component of $\vec{v}$, scalar v_2 is the j-component of $\vec{v}$, and scalar v_3 is the k-component of $\vec{v}$,

$\mathbb{R}^2$. Similarily, in $\mathbb{R}^2$, the standard unit vectors are:

$$\vec{\imath} \stackrel{\text{def}}{=} \langle 1,0\rangle \qquad \text{and} \qquad \vec{\jmath} \stackrel{\text{def}}{=} \langle 0,1\rangle .$$

and any vector in $\mathbb{R}^2$ is a linear combination of the standard unit vectors: $\vec{v} = \langle v_1, v_2\rangle = v_1\,\vec{\imath} + v_2\,\vec{\jmath}$.

9. If $|\vec{v}| \neq 0$, it is often helpful to write/think-of $\vec{v}$ as:

$$\vec{v} = \underbrace{|\vec{v}|}_{\text{the length of } \vec{v}} \; \underbrace{\frac{\vec{v}}{|\vec{v}|}}_{\boxed{\text{a unit vector in the direction of } \vec{v}}} \tag{3}$$

Ex. Find the direction of the vector $\overrightarrow{AB}$ where the point $A = (1,2,3)$ and the point $B = (6,5,4)$. Express your answer as a linear combination of the standard unit vectors.

Soln: The direction of $\overrightarrow{AB}$ is $\boxed{\qquad\qquad\qquad\qquad}$.

The direction of the vector $\overrightarrow{AB}$ is the unit vector in the same direction as $\overrightarrow{AB}$, which is $\frac{\overrightarrow{AB}}{|\overrightarrow{AB}|}$ by (3).

Note $\overrightarrow{AB} = \langle 6-1\ ,\ 5-2\ ,\ 4-3\rangle = \langle 5,3,1\rangle$. So $\left|\overrightarrow{AB}\right| = \sqrt{5^2+3^2+1^3} = \sqrt{25+9+1} = \sqrt{35}$.

So $\dfrac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|} = \dfrac{\langle 5,3,1\rangle}{\sqrt{35}} = \dfrac{1}{\sqrt{35}}\langle 5,3,1\rangle = \left\langle \dfrac{5}{\sqrt{35}}, \dfrac{3}{\sqrt{35}}, \dfrac{1}{\sqrt{35}}\right\rangle = \dfrac{5}{\sqrt{35}}\vec{\imath} + \dfrac{3}{\sqrt{35}}\vec{\jmath} + \dfrac{1}{\sqrt{35}}\vec{k}$

10. Properties of Vector Operations.

Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors. Let a and b be scalars (i.e., real numbers).

(1) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

(2) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

(3) $\vec{u} + \vec{0} = \vec{u}$

(4) $\vec{u} + (-\vec{u}) = \vec{0}$

and

(5) $0\,\vec{u} = \vec{0}$

(6) $1\,\vec{u} = \vec{u}$

(7) $a\,(b\,\vec{u}) = (ab)\,\vec{u}$

(8) $a\,(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$

(9) $(a+b)\,\vec{u} = a\vec{u} + b\vec{u}$