CHAPTER 37

Parametric Representation of Curves

Parametric Equations

If the coordinates (x, y) of a point P on a curve are given as functions $x = f(u)$, $y = g(u)$ of a third variable or *parameter*, u, the equations $x = f(u)$ and $y = g(u)$ are called *parametric equations* of the curve.

EXAMPLE 37.1:

(a) $x = \cos\theta$, $y = 4\sin^2\theta$ are parametric equations, with parameter θ, of the parabola $4x^2 + y = 4$, since $4x^2 + y = 4\cos^2\theta + 4\sin^2\theta = 4$.

(b) $x = \frac{1}{2}t$, $y = 4 - t^2$ is another parametric representation, with parameter t, of the same curve.

It should be noted that the first set of parametric equations represents only a portion of the parabola (Fig. 37-1(*a*)), whereas the second represents the entire curve (Fig. 37-1(*b*)).

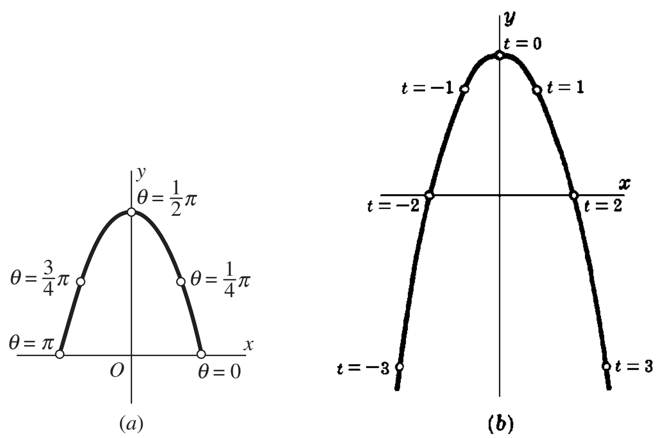

Fig. 37-1

EXAMPLE 37.2:

(a) The equations $x = r\cos\theta$, $y = r\sin\theta$ represent the circle of radius r with center at the origin, since $x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2$. The parameter θ can be thought of as the angle from the positive x axis to the segment from the origin to the point P on the circle (Fig. 37-2).

(b) The equations $x = a + r\cos\theta$, $y = b + r\sin\theta$ represents the circle of radius r with center at (a, b), since $(x - a)^2 + (y - b)^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2$.

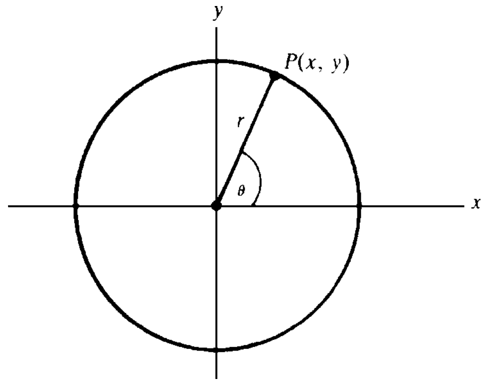

Fig. 37-2

Assume that a curve is specified by means of a pair of parametric equations $x = f(u)$ and $y = g(u)$. Then the first and second derivatives $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are given by the following formulas.

(37.1) First Derivative

$$\frac{dy}{dx} = \left(\frac{dy}{du}\right) \bigg/ \left(\frac{dx}{du}\right)$$

This follows from the Chain Rule formula $\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du}$.

(37.2) Second Derivative

$$\frac{d^2y}{dx^2} = \left(\frac{d}{du}\left(\frac{dy}{dx}\right)\right) \bigg/ \frac{dx}{du}$$

This follows from the Chain Rule formula $\frac{d}{du}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} \cdot \frac{dx}{du}$.

Arc Length for a Parametric Curve

If a curve is given by parametric equations $x = f(t)$, $y = g(t)$, then the length of the arc of the curve between the points corresponding to parameter values t_1 and t_2 is

$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

This formula can be derived by an argument similar to that for the arc length formula (29.2).

SOLVED PROBLEMS

1. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if $x = t - \sin t$, $y = 1 - \cos t$.

$\frac{dx}{dt} = 1 - \cos t$ and $\frac{dy}{dt} = \sin t$. By (37.1), $\frac{dy}{dx} = \frac{\sin t}{1 - \cos t}$. Then

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{(1 - \cos t)(\cos t) - (\sin t)(\sin t)}{(1 - \cos t)^2}$$

$$= \frac{\cos t - (\cos^2 t + \sin^2 t)}{(1 - \cos t)^2} = \frac{\cos t - 1}{(1 - \cos t)^2} = \frac{1}{\cos t - 1}$$

Hence, by (37.2),

$$\frac{d^2y}{dx^2} = \frac{1}{\cos t - 1}\Big/(1 - \cos t) = -\frac{1}{(1-\cos t)^2}$$

2. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if $x = e^t \cos t$, $y = e^t \sin t$.

$\frac{dx}{dt} = e^t(\cos t - \sin t)$ and $\frac{dy}{dt} = e^t(\cos t + \sin t)$. By (37.1), $\frac{dy}{dx} = \frac{\cos t + \sin t}{\cos t - \sin t}$. Then,

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{(\cos t - \sin t)^2 - (\cos t + \sin t)(-\sin t - \cos t)}{(\cos t - \sin t)^2}$$

$$= \frac{(\cos t - \sin t)^2 + (\cos t + \sin t)^2}{(\cos t - \sin t)^2} = \frac{2(\cos^2 t + \sin^2 t)}{(\cos t - \sin t)^2}$$

$$= \frac{2}{(\cos t - \sin t)^2}$$

So, by (37.2),

$$\frac{d^2y}{dx^2} = \frac{2}{(\cos t - \sin t)^2}\Big/e^t(\cos t - \sin t) = \frac{2}{e^t(\cos t - \sin t)}$$

3. Find an equation of the tangent line to the curve $x = \sqrt{t}$, $y = t - \frac{1}{\sqrt{t}}$ at the point where $t = 4$.

$\frac{dx}{dt} = \frac{1}{2\sqrt{t}}$ and $\frac{dy}{dt} = 1 + \frac{1}{2t^{3/2}}$. By (37.1), $\frac{dy}{dx} = 2\sqrt{t} + \frac{1}{t}$. So, the slope of the tangent line when $t = 4$ is $2\sqrt{4} + \frac{1}{4} = \frac{17}{4}$. When $t = 4$, $x = 2$ and $y = \frac{7}{2}$. An equation of the tangent line is $y - \frac{7}{2} = \frac{17}{4}(x-2)$.

4. The position of a particle that is moving along a curve is given at time t by the parametric equations $x = 2 - 3\cos t$, $y = 3 + 2\sin t$, where x and y are measured in feet and t in seconds. (See Fig. 37-3.) Note that $\frac{1}{9}(x-2)^2 + \frac{1}{4}(y-3)^2 = 1$, so that the curve is an ellipse. Find: (a) the time rate of change of x when $t = \pi/3$; (b) the time rate of change of y when $t = 5\pi/3$; (c) the time rate of change of the angle of inclination θ of the tangent line when $t = 2\pi/3$.

$\frac{dy}{dt} = 3\sin t$ and $\frac{dy}{dt} = 2\cos t$. Then $\tan\theta = \frac{dy}{dx} = \frac{2}{3}\cot t$.

(a) When $t = \frac{\pi}{3}$, $\frac{dx}{dt} = \frac{3\sqrt{3}}{2}$ ft/sec

(b) When $t = \frac{5\pi}{3}$, $\frac{dy}{dt} = 2(\frac{1}{2}) = 1$ ft/sec

(c) $\theta = \tan^{-1}(\frac{2}{3}\cot t)$. So, $\frac{d\theta}{dt} = \frac{-\frac{2}{3}\csc^2 t}{1 + \frac{4}{9}\cot t^2\, t} = \frac{-6\csc^2 t}{9 + 4\cot^2 t}$.

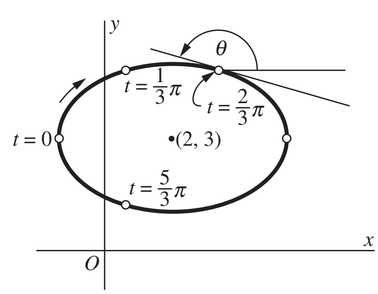

Fig. 37-3

When $t = \frac{2\pi}{3}$, $\frac{d\theta}{dt} = \frac{-6(2/\sqrt{3})^2}{9+4(-1/\sqrt{3})^2} = -\frac{24}{31}$. Thus, the angle of inclination of the tangent line is decreasing at the rate of $\frac{24}{31}$ radians per second.

5. Find the arc length of the curve $x = t^2$, $y = t^3$ from $t = 0$ to $t = 4$.

$\frac{dx}{dt} = 2t$, $\frac{dy}{dt} = 3t^2$ and $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4t^2 + 9t^4 = 4t^2(1+\frac{9}{4}t^2)$.

Then

$$L = \int_0^4 2t\sqrt{1+\tfrac{9}{4}t^2}\,dt = \tfrac{4}{9}\int_0^4 (1+\tfrac{9}{4}t^2)^{1/2}(\tfrac{9}{2}t)\,dt$$

$$= \tfrac{4}{9}\tfrac{2}{3}(1+\tfrac{9}{4}t^2)^{3/2}\Big]_0^4 = \tfrac{8}{27}(37\sqrt{37}-1)$$

6. Find the length of an arch of the cycloid $x = \theta - \sin\theta$, $y = 1 - \cos\theta$ between $\theta = 0$ and $\theta = 2\pi$.

$\frac{dx}{d\theta} = 1-\cos\theta$, $\frac{dy}{d\theta} = \sin\theta$ and $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (1-\cos\theta)^2 + \sin^2\theta = 2(1-\cos\theta) = 4\sin^2\left(\frac{\theta}{2}\right)$. Then

$$L = 2\int_0^4 \sin\left(\frac{\theta}{2}\right)d\theta = -4\cos\left(\frac{\theta}{2}\right)\Big]_0^{2\pi} = -4(\cos\pi - \cos 0) = 8$$

SUPPLEMENTARY PROBLEMS

In Problems 7–11, find: (*a*) $\frac{dy}{dx}$; (*b*) $\frac{d^2y}{dx^2}$.

7. $x = 2 + t$, $y = 1 + t^2$ *Ans.* (a) $2t$; (b) 2

8. $x = t + 1/t$, $y = t + 1$ *Ans.* (a) $t^2/(t^2-1)$; (b) $-2t^3/(t^2-1)^3$

9. $x = 2\sin t$, $y = \cos 2t$ *Ans.* (a) $-2\sin t$; (b) -1

10. $x = \cos^3\theta$, $y = \sin^3\theta$ *Ans.* (a) $-\tan\theta$; (b) $1/(3\cos^4\theta\sin\theta)$

11. $x = a(\cos\phi + \phi\sin\phi)$, $y = a(\sin\phi - \phi\cos\phi)$ *Ans.* (a) $\tan\phi$; (b) $1/(a\phi\cos^3\phi)$

12. Find the slope of the curve $x = e^{-t}\cos 2t$, $y = e^{-2t}\sin 2t$ at the point $t = 0$.

Ans. -2

13. Find the rectangular coordinates of the highest point of the curve $x = 96t$, $y = 96t - 16t^2$. (*Hint:* Find t for maximum y.)

Ans. (288, 144)

14. Find equations of the tangent line and normal line to the following curves at the points determined by the given value of the parameter:

(a) $x = 3e^t$, $y = 5e^{-t}$ at $t = 0$
(b) $x = a\cos^4\theta$, $y = a\sin^4\theta$ at $\theta = \frac{\pi}{4}$

Ans. (a) $3y + 5x = 30$, $5y - 3x = 16$; (b) $2x + 2y = a$, $y = x$

15. Find an equation of the tangent line at any point $P(x, y)$ of the curve $x = a\cos^3 t$, $y = a\sin^3 t$. Show that the length of the segment of the tangent line intercepted by the coordinate axes is a.

Ans. $x\sin t + y\cos t = \frac{a}{2}\sin 2t$

16. For the curve $x = t^2 - 1$, $y = t^3 - t$, locate the points where the tangent line is (a) horizontal, and (b) vertical. Show that, at the point where the curve crosses itself, the two tangent lines are mutually perpendicular.

Ans. (a) $t = \pm\frac{\sqrt{3}}{3}$; (b) $t = 0$

In Problems 17–20, find the length of the specified arc of the given curve.

17. The circle $x = a\cos\theta$, $y = a\sin\theta$ from $\theta = 0$ to $\theta = 2\pi$.

Ans. $2\pi a$

18. $x = e^t\cos t$, $y = e^t\sin t$ from $t = 0$ to $t = 4$.

Ans. $\sqrt{2}(e^4 - 1)$

19. $x = \ln\sqrt{1+t^2}$, $y = \tan^{-1} t$ from $t = 0$ to $t = 1$.

Ans. $\ln(1+\sqrt{2})$

20. $x = 2\cos\theta + \cos 2\theta + 1$, $y = 2\sin\theta + \sin 2\theta$.

Ans. 16

21. The position of a point at time t is given as $x = \frac{1}{2}t^2$, $y = \frac{1}{9}(6t+9)^{3/2}$. Find the distance the point travels from $t = 0$ to $t = 4$.

Ans. 20

22. Identify the curves given by the following parametric equations and write equations for the curves in terms of x and y:

(a) $x = 3t + 5$, $y = 4t - 1$ *Ans.* Straight line: $4x - 3y = 23$
(b) $x = t + 2$, $y = t^2$ *Ans.* Parabola: $y = (x - 2)^2$
(c) $x = t - 2$, $y = \frac{t}{t-2}$ *Ans.* Hyperbola: $y = \frac{2}{x} + 1$
(d) $x = 5\cos t$, $y = 5\sin t$ *Ans.* Circle: $x^2 + y^2 = 25$

23. (GC) Use a graphing calculator to find the graphs of the following parametric curves:

(a) $x = \theta + \sin\theta$, $y = 1 - \cos\theta$ (cycloid)
(b) $x = 3\cos^3\theta$, $y = 3\sin^3\theta$ (hypocycloid)
(c) $x = 2\cot\theta$, $y = 2\sin^2\theta$ (witch of Agnesi)
(d) $x = \frac{3\theta}{(1+\theta^3)}$, $y = \frac{3\theta^2}{(1+\theta^3)}$ (folium of Descartes)