NAME: ______________________________ **PIN**: ______________

1. Fill in the two blanks. By using the Limit Comparison Test, one can show that the formal series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^3+100n^2+5n-3}}{1002n^4+n-1}\,. \tag{1}$$

is ______________ by comparing the series in (1) to the p-series $\sum\left(\frac{1}{n}\right)^p$ with ________ .

a. convergent, $p=\frac{5}{2}$
b. divergent, $p=\frac{5}{2}$
c. convergent, $p=1$
d. divergent, $p=1$
e. none of the others

First let's recall the **Limit Comparison Test** for a positive-termed series $\sum a_n$.
Let $b_n>0$ and $L=\lim_{n\to\infty}$ $\boxed{\frac{a_n}{b_n}}$.

- If $\boxed{0<L<\infty}$, then $\boxed{[\ \sum b_n \text{ converges} \iff \sum a_n \text{ converges}\ ]}$.
- If $\boxed{L=0}$, then $\boxed{[\ \sum b_n \text{ converges} \implies \sum a_n \text{ converges}\ ]}$.
- If $\boxed{L=\infty}$, then $\boxed{[\ \sum b_n \text{ diverges} \implies \sum a_n \text{ diverges}\ ]}$.

To apply the Limit Comparison Test to the problem given, let

$$a_n = \frac{\sqrt{n^3+100n^2+5n-3}}{1002n^4+n-1}$$

and we are told to let $b_n=\frac{1}{n^p}$ for some p (we need to decide what p). Since

$$\sum_{n=1}^{\infty}\frac{1}{n^p} \quad \text{is} \quad \begin{cases}\text{convergent} & \text{if } p=\frac{5}{2} \text{ (since } p>1)\\ \text{divergent} & \text{if } p=1 \text{ (since } p\le 1)\end{cases}$$

we know choice **b** and **c** cannot be correct. For n sufficiently big,

$$a_n = \frac{\sqrt{n^3+100n^2+5n-3}}{1002n^4+n-1} \approx \frac{\sqrt{n^3}}{1002n^4} = \frac{1}{1002}\frac{n^{3/2}}{n^{8/2}} = \frac{1}{1002}\frac{1}{n^{5/2}}\,.$$

So we let $b_n=\left(\frac{1}{n}\right)^{\frac{5}{2}}$ and compute

$$\frac{a_n}{b_n} = \frac{\left(n^3+100n^2+5n-3\right)^{1/2}}{1002n^4+n-1}\,\frac{\left(n^5\right)^{1/2}}{1} = \left[\frac{n^8+100n^7+5n^6-3n^5}{\left(1002n^4+n-1\right)^2}\right]^{1/2} \xrightarrow{n\to\infty} \left[\frac{1}{(1002)^2}\right]^{1/2} = \frac{1}{1002}\,.$$

Note that we used that $\left(1002n^4+n-1\right)^2$ is a polynomial of degree 8 and the coefficient of n^8 is $(1002)^2$. Since $\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{1}{1002}$ and $0<\frac{1}{1002}<\infty$, the LCT says that the $\sum a_n$ and $\sum b_n$ *do the some thing*. Since we know that $\sum b_n$ converges (it is a p-series with $p=5/2>1$), the LCT says that $\sum a_n$ also converges.

So the solution is **a**.