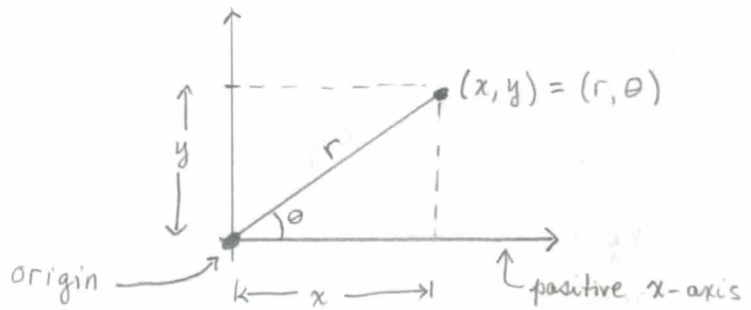
Polar Coordinates

Cartesian Coordinates (x, y) ← unique representation of a point

vs. Polar Coordinates (r, θ) ← there are many representation of the same point.

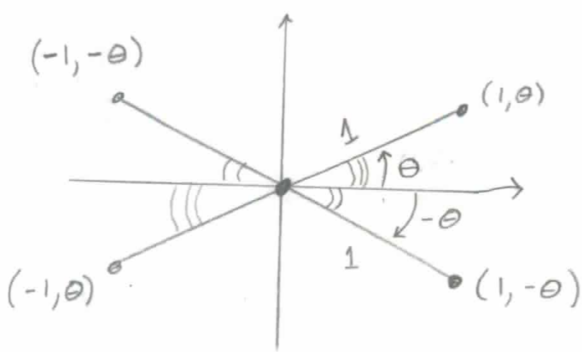


- See handout – conversion
- all should be clear in the 1st Quadrant, you should verify the other quadrants.

Ex 1 $(1, \theta)$ vs. $(1, -\theta)$ vs $\underbrace{(-1, \theta)}$ vs $\underbrace{(-1, -\theta)}$

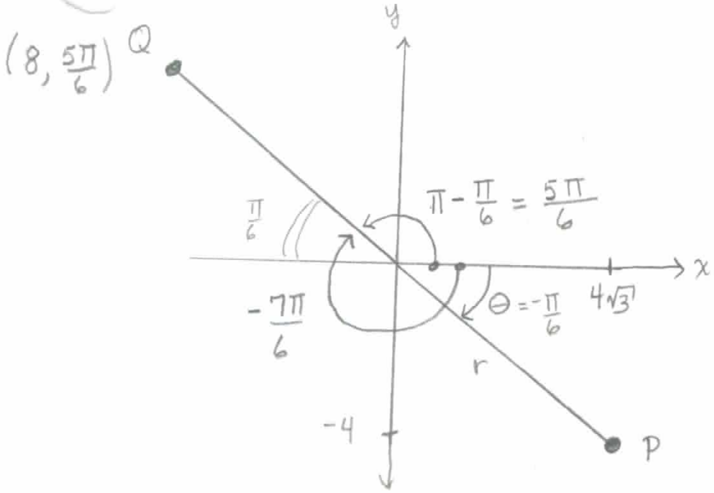
↳ reflect $(1,\theta)$ over origin. ↳ reflect $(1,-\theta)$ over origin

picture for when $0 < \theta < \frac{\pi}{2}$



Ex 2 In Cartesian Coordinates $P = (4\sqrt{3}, -4)$

In Polar Coordinates $P = (r, \theta) = ??$



① $r^2 = x^2 + y^2 = (4\sqrt{3})^2 + (-4)^2$

$= 16 \cdot 3 + 16 = 16(3+1) = 16 \cdot 4$

$\Rightarrow r = \sqrt{16 \cdot 4} = 4 \cdot 2 = 8$

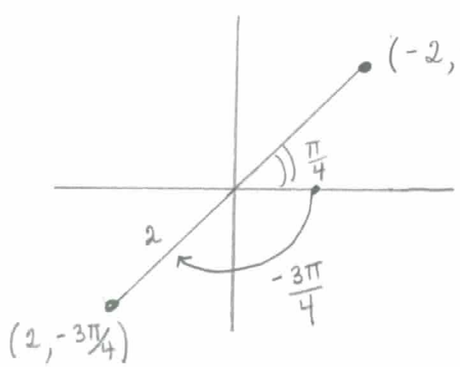
② $\tan\theta = \frac{y}{x} = \frac{-4}{4\sqrt{3}} = -\frac{1}{\sqrt{3}}$

$\Rightarrow \theta = -\frac{\pi}{6}$

So $P = (8, -\frac{\pi}{6}) = (8, \underset{2\pi + -\pi/6}{\underset{\parallel}{\frac{11\pi}{6}}}) = \underbrace{(-8, \frac{5\pi}{6})}_{\text{look at } Q} = \underbrace{(-8, -\frac{7\pi}{6})}_{Q = (8, \frac{5\pi}{6}) = (8, -\frac{7\pi}{6})}$

Ex 3. In Polar Coords $P = (-2, -\frac{3\pi}{4})$

In Cartesian Coords $P = (x, y) = ??$ ← only one answer



← note, $P = (-2, -\frac{3\pi}{4}) = (2, \frac{\pi}{4})$

For P:

$x = r\cos\theta = -2\cos(-\frac{3\pi}{4}) = -2(-\frac{\sqrt{2}}{2}) = \sqrt{2}$

$y = r\sin\theta = -2\sin(-\frac{3\pi}{4}) = -2(-\frac{\sqrt{2}}{2}) = \sqrt{2}$

So $P = (\sqrt{2}, \sqrt{2})$

Ex. 4 Convert the Cartesian equation $xy = 1$ into a polar equations.

$$xy = 1 \iff (r\cos\theta)(r\sin\theta) = 1$$

$$\iff r^2 \cos\theta \sin\theta = 1$$

← ok, but can you do better (to help with graphing) try to write it using only one trig. function.

$$\iff \frac{r^2}{2}(2\cos\theta\sin\theta) = 1$$

$$\iff \frac{r^2}{2}(\sin 2\theta) = 1$$

$$\iff \boxed{r^2 \sin 2\theta = 2}$$

Ex. 5. Convert the polar equation $\boxed{r = 2\sin\theta}$ into a Cartesion equation.

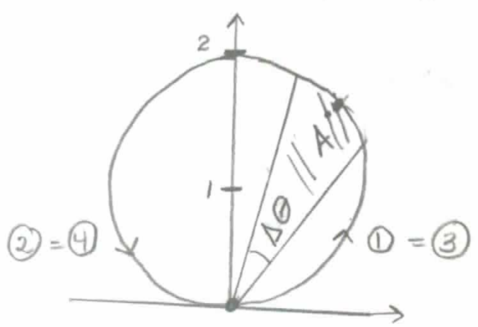
$$r = 2\sin\theta \iff r^2 = 2r\sin\theta$$

$$\iff x^2 + y^2 = 2y$$

← ok, but can you do better? what is it?

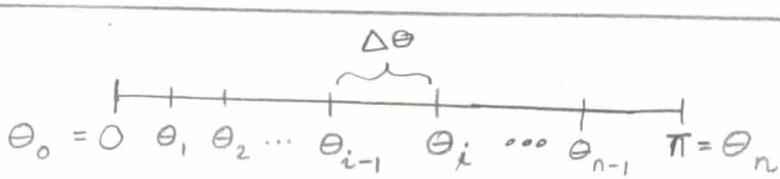
$$\iff x^2 + y^2 - 2y + 1 = 1$$

$$\iff x^2 + (y-1)^2 = 1.$$ ← circle with $\begin{cases} \text{radius} = 1 \\ \text{center} = (0, 1) \end{cases}$



Let's look closer

θ	$\sin\theta$	$r = 2\sin\theta$
$0 \xrightarrow{①} \frac{\pi}{2}$	$0 \to 1$	$0 \to 2$
$\frac{\pi}{2} \xrightarrow{②} \pi$	$1 \to 0$	$2 \to 0$
$\pi \xrightarrow{③} \frac{3\pi}{2}$	$0 \to -1$	$0 \to -2$
$\frac{3\pi}{2} \xrightarrow{④} 2\pi$	$-1 \to 0$	$-2 \to 0$



Let's talk area

Area of typical sector $= \frac{1}{2} r^2 \theta = \frac{1}{2}(2\sin\theta)^2 \Delta\theta = 2\sin^2\theta\, \Delta\theta$

Area of circle $= \int_{\theta=0}^{\theta=\pi} (2\sin^2\theta)\, d\theta \overset{\text{trig ID}}{=} \int_{\theta=0}^{\theta=\pi} (1 - \cos 2\theta)\, d\theta \overset{\text{Calc.}}{=} \pi.$ ← why?

2 → Let's look at Polar Coordinate handout

- Area $A = \frac{1}{2}\int_{\theta=\alpha}^{\theta=\beta} r^2\, d\theta = \frac{1}{2}\int_{\theta=\alpha}^{\theta=\beta} [f(\theta)]^2\, d\theta$

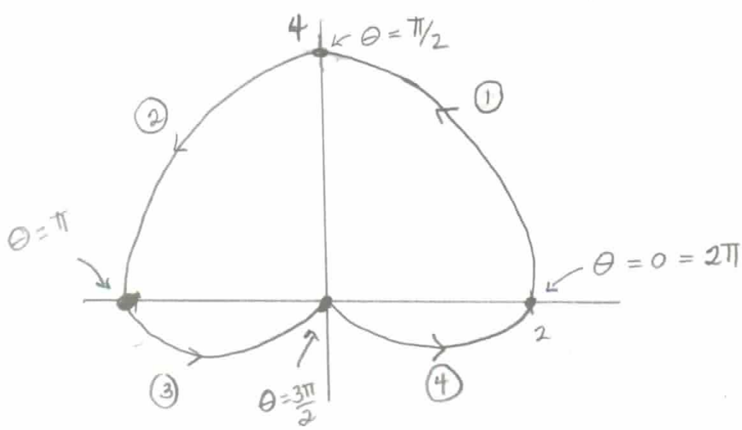
- Graphing

So in Ex 5 with $r = 2\sin\theta$: $\begin{cases} \text{period} = \frac{2\pi}{1} = 2\pi \\ \frac{\text{period}}{4} = \frac{2\pi}{4} = \frac{\pi}{2} \end{cases}$

Ex 6. $\boxed{r = 2 + 2\sin\theta}$

(6a) Sketch the graph . period of $(\sin\theta = \sin 1\theta) = \frac{2\pi}{1} = 2\pi$. $\Rightarrow \frac{\text{period}}{4} = \frac{2\pi}{4} = \frac{\pi}{2}$.

θ	$\sin\theta$	$2\sin\theta$	$r = 2 + 2\sin\theta$
① $0 \to \frac{\pi}{2}$	$0 \to 1$	$0 \to 2$	$2 \to 4$
② $\frac{\pi}{2} \to \pi$	$1 \to 0$	$2 \to 0$	$4 \to 2$
③ $\pi \to \frac{3\pi}{2}$	$0 \to -1$	$0 \to -2$	$2 \to 0$
④ $\frac{3\pi}{2} \to 2\pi$	$-1 \to 0$	$-2 \to 0$	$0 \to 2$



(6b) Express the area A enclosed by $r = 2 + 2\sin\theta$ as an integral with respect to θ.

$$A = \frac{1}{2}\int_{\theta=\alpha}^{\theta=\beta} r^2\, d\theta = \frac{1}{2}\int_{\theta=0}^{\theta=2\pi} (2+2\sin\theta)^2\, d\theta$$

(or by symmetry) $= 2 \cdot \frac{1}{2}\int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} (2+2\sin\theta)^2\, d\theta$

(6c) Express the length L of the curve traced by $\boxed{r = 2 + 2\sin\theta}$ as an integral with respect to θ.

$$L = \int_{\theta=\alpha}^{\theta=\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \boxed{\int_{\theta=0}^{\theta=2\pi} \sqrt{(2+2\sin\theta)^2 + (2\cos\theta)^2}\, d\theta}$$

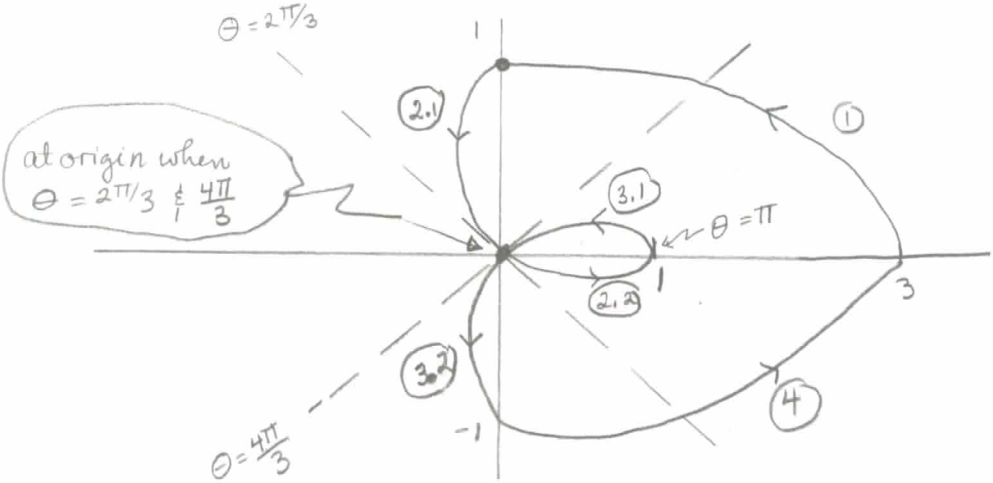
Ex 7 $\boxed{r = 1 + 2\cos\theta}$

(7a) Sketch the graph $\dfrac{\text{period of } (\cos\theta)}{4} = \dfrac{2\pi}{4} = \dfrac{\pi}{2}$

	θ	$\cos\theta$	$r = 1+2\cos\theta$		θ	$\cos\theta$	r
1	$0 \to \frac{\pi}{2}$	$1 \to 0$	$3 \to 1$	2.1	$\frac{\pi}{2} \to \frac{2\pi}{3}$	$0 \to -\frac{1}{2}$	$1 \to 0$
2	$\frac{\pi}{2} \to \pi$	$0 \to -1$	$1 \to -1$	2.2	$\frac{2\pi}{3} \to \pi$	$-\frac{1}{2} \to -1$	$0 \to -1$
3	$\pi \to \frac{3\pi}{2}$	$-1 \to 0$	$-1 \to 1$	3.1	$\pi \to \frac{4\pi}{3}$	$-1 \to -\frac{1}{2}$	$-1 \to 0$
4	$\frac{3\pi}{2} \to 2\pi$	$0 \to 1$	$1 \to 3$	3.2	$\frac{4\pi}{3} \to \frac{3\pi}{2}$	$-\frac{1}{2} \to 0$	$0 \to 1$

here r passes thru zero so split up interval of θ for when $r=0$

$r = 0 \iff 0 = 1 + 2\cos\theta \iff \cos\theta = -\frac{1}{2} \overset{0 \le \theta \le 2\pi}{\iff} \theta = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$.



(7b) Express the area A inside the big loop, outside the little loop, using integral(s) w.r.t. θ.

A = 2 [the part of the desired area in the 1st and 2nd Quad.]

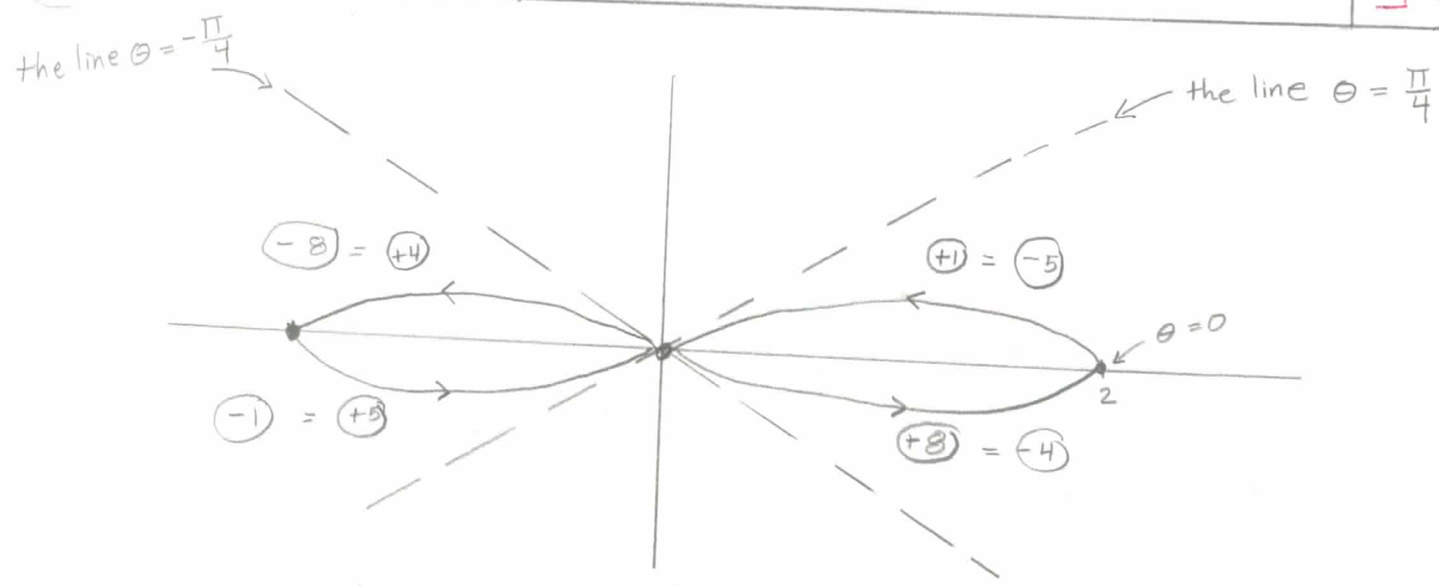
$$= 2\Bigg[\underbrace{\frac{1}{2}\int_{\theta=0}^{\theta=\frac{2\pi}{3}} (1+2\cos\theta)^2\,d\theta}_{\text{from 1 and 2.1}} \underset{\text{minus}}{-} \underbrace{\frac{1}{2}\int_{\theta=\pi}^{\theta=\frac{4\pi}{3}} (1+2\cos\theta)^2\,d\theta}_{\text{from 3.1}}\Bigg]$$

Ex 8 $\boxed{r^2 = 4\cos(2\theta)}$

(8a) Sketch the graph   period of $(\cos 2\theta) = \frac{2\pi}{2} = \pi \Rightarrow \frac{\text{period}}{4} = \frac{\pi}{4}$

	θ	2θ	$\cos(2\theta)$	$r^2 = 4\cos(2\theta)$	$r = +\sqrt{r^2}$	$r = -\sqrt{r^2}$
1st Q	① $0 \to \frac{\pi}{4}$	$0 \to \frac{\pi}{2}$	$1 \to 0$	$4 \to 0$	$2 \to 0$	$-2 \to 0$
	② $\frac{\pi}{4} \to \frac{\pi}{2}$	$\frac{\pi}{2} \to \pi$	$0 \to -1$	$0 \to -4$	no graph	no graph
2nd Q	③ $\frac{\pi}{2} \to \frac{3\pi}{4}$	$\pi \to \frac{3\pi}{2}$	$-1 \to 0$	$-4 \to 0$	no graph	no graph
	④ $\frac{3\pi}{4} \to \pi$	$\frac{3\pi}{2} \to 2\pi$	$0 \to 1$	$0 \to 4$	$0 \to 2$	$0 \to -2$
3rd Q	⑤ $\pi \to \frac{5\pi}{4}$	$2\pi \to$	$1 \to 0$	$4 \to 0$	$2 \to 0$	$-2 \to 0$
	⑥ $\frac{5\pi}{4} \to \frac{3\pi}{2}$	just go around unit circle again - so fake	$0 \to -1$	$0 \to -4$	no graph	no graph
4th Q	⑦ $\frac{3\pi}{2} \to \frac{7\pi}{4}$		$-1 \to 0$	$-4 \to 0$	no graph	no graph
	⑧ $\frac{7\pi}{4} \to 2\pi$		$0 \to 1$	$0 \to 4$	$0 \to 2$	$0 \to -2$

Lesson: the graph of $(-\sqrt{r^2}, \theta)$ is the reflection across the origin of the graph of $(+\sqrt{r^2}, \theta)$

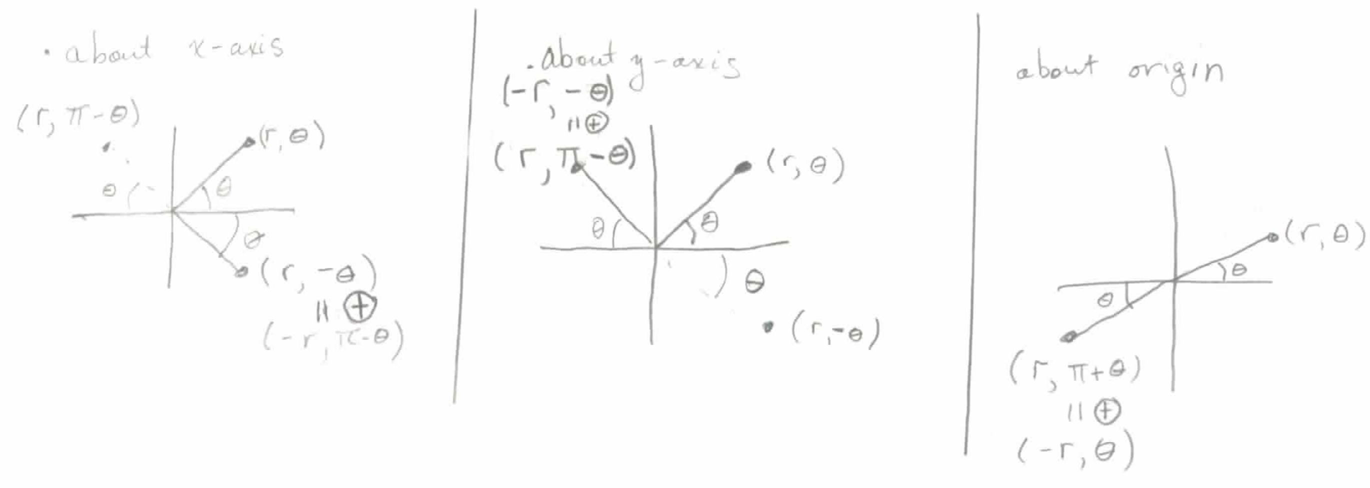


(8b) Express the area A enclosed by $r^2 = 4\cos(2\theta)$ as an integral with respect to θ.

$$A = \frac{1}{2}\int_{\theta=\alpha}^{\theta=\beta} r^2\, d\theta \underset{\text{symmetry}}{=} 4 \cdot \underbrace{\frac{1}{2}\int_{\theta=0}^{\theta=\pi/4} 4\cos(2\theta)\, d\theta}_{\text{area in 1st Quad}}$$

$$= \boxed{8\int_{\theta=0}^{\theta=\pi/4} \cos(2\theta)\, d\theta}$$

Symmetry ⊕ Recall, $(-r,\theta)$ represents same point as $(r, \theta+\pi)$ + see handout/Polar Coordinates



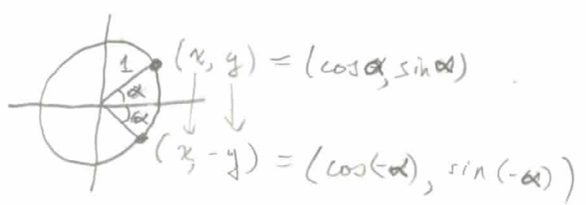
If a substitution listed below does not change the sol'n to a polar equation, then have symmetry:

substitution for (r,θ)	symmetry
$(r,-\theta)$	x axis
$(-r, \pi-\theta)$	x axis
$(-r,-\theta)$	y axis
$(r, \pi-\theta)$	y axis
$(-r,\theta)$	origin
$(r, \pi+\theta)$	origin

* do not memorize chart
- be able to reproduce picture above

Because of the multiple representation of pts in polar cords, symmetries may exist that are not identified by these tests.

Recall



$\cos(-\alpha) = +\cos(\alpha)$

$\sin(-\alpha) = -\sin(\alpha)$

Ex 9 Discuss the symmetry of the graph of

$$\boxed{r = 3\cos(2\theta) \quad (*)}$$

(1) Symmetric about x-axis?

- In (*), replace (r, θ) with $(r, -\theta)$

(*) $r = 3\cos(2\theta)$ the same so yes

$r = 3\cos(2(-\theta)) \iff r = 3\cos(-2\theta) \iff r = 3\cos(2\theta)$

b/c $\cos(-\alpha) = \cos(\alpha)$

(2) symmetric about y-axis?

- In (*), replace (r, θ) with $(-r, -\theta)$

(*) $r = 3\cos(2\theta)$ not the same so maybe

$-r = 3\cos(2(-\theta)) \iff -r = 3\cos(-2\theta) \iff -r = 3\cos 2\theta$

b/c $\cos(-\alpha) = \cos\alpha$

(3) symmetric about y-axis?

- In (*), replace (r, θ) with $(r, \pi - \theta)$

(*) $r = 3\cos(2\theta)$ the same so yes

$r = 3\cos(2(\pi - \theta)) \iff r = 3\cos(2\pi - 2\theta) \iff r = 3\cos(2\theta)$

b/c $\cos(2\pi - 2\theta) = \cos(-2\theta) = \cos(2\theta)$

(4) symmetric about the origin?

In (*), replace (r, θ) with $(-r, \theta)$

(*) $r = 3\cos(2\theta)$ not the same so maybe

$-r = 3\cos(2\theta)$

(5) symmetric about the origin?

In (*), replace (r, θ) with $(r, \pi + \theta)$

(*) $r = 3\cos(2\theta)$ the same so yes

$r = 3\cos(2(\pi + \theta)) \iff r = 3\cos(2\pi + 2\theta) \iff r = 3\cos 2\theta$

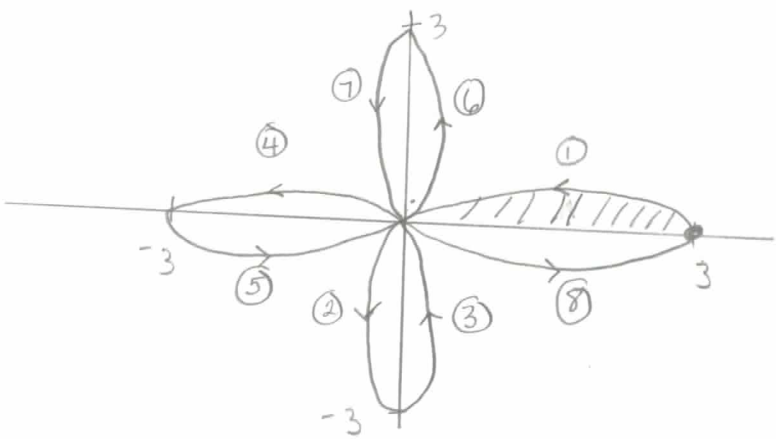
b/c $\cos(2\pi + \alpha) = \cos\alpha$.

Ex 10 $\boxed{r = 3\cos 2\theta}$

(10a) Sketch the graph

$$\frac{\text{period of } \cos(2\theta)}{4} = \frac{\frac{2\pi}{2}}{4} = \frac{\pi}{4}$$

	θ	2θ	$\cos(2\theta)$	$r = 3\cos(2\theta)$
1	$0 \to \frac{\pi}{4}$	$0 \to \frac{\pi}{2}$	$1 \to 0$	$3 \to 0$
2	$\frac{\pi}{4} \to \frac{\pi}{2}$	$\frac{\pi}{2} \to \pi$	$0 \to -1$	$0 \to -3$
3	$\frac{\pi}{2} \to \frac{3\pi}{4}$	$\pi \to \frac{3\pi}{2}$	$-1 \to 0$	$-3 \to 0$
4	$\frac{3\pi}{4} \to \pi$	$\frac{3\pi}{2} \to 2\pi$	$0 \to 1$	$0 \to 3$
5	$\pi \to \frac{5\pi}{4}$	$2\pi \to$	$1 \to 0$	$3 \to 0$
6	$\frac{5\pi}{4} \to \frac{3\pi}{2}$	just go around unit circle again so fake	$0 \to -1$	$0 \to -3$
7	$\frac{3\pi}{2} \to \frac{7\pi}{4}$		$-1 \to 0$	$-3 \to 0$
8	$\frac{7\pi}{4} \to 2\pi$		$0 \to 1$	$0 \to 3$



P.S: You do not have to know the names (eg: Limaçon, Cardioid) of the various polar graphs

(10b) Express the area A of the enclosed region as an integral with respect to θ

Area of enclosed region, by symmetry.

$= 8$ (shaded area above)

$$= 8 \cdot \frac{1}{2} \int_{\theta=\alpha}^{\theta=\beta} r^2 \, d\theta = \boxed{8 \cdot \frac{1}{2} \int_{\theta=0}^{\theta=\frac{\pi}{4}} (3\cos 2\theta)^2 \, d\theta}$$

or

$$= 36 \int_{\theta=0}^{\theta=\pi/4} \cos^2(2\theta)\, d\theta$$