The plane (i.e., the 2-dimensional space) $\mathbb{R}^2$ is $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) : x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$.

Definition

Given:

(1) an interval $I \subset \mathbb{R}$
(2) a function $f\colon I \to \mathbb{R}$
(3) another function $g\colon I \to \mathbb{R}$.

Then we form

(4) the function $h\colon I \to \mathbb{R}^2$ by letting $h(t) = (f(t), g(t))$ for $t \in I$.

So for a fixed number $t_0 \in I \subset \mathbb{R}$, the point $h(t_0) = (f(t_0), g(t_0)) \in \mathbb{R}^2$. We call

$$\mathcal{C} = \left\{(f(t), g(t)) \in \mathbb{R}^2 : t \in I\right\}$$

a paramteric (planar) curve, which is parametrized by the functions f and g. Often we write as

$$\begin{aligned} x &= f(t) \\ y &= g(t) \end{aligned} \quad , t \in I$$

or write as

$$\begin{aligned} x &= x(t) \\ y &= y(t) \end{aligned} \quad , t \in I$$

►. We think of t as time and $\mathcal{C}$ describing the motion of a puffo as he moves through the plane $\mathbb{R}^2$.

Calculus with Parametric Curves

Consider the curve $\mathcal{C}$ parameterized by

$$\begin{aligned} x &= x(t) \\ y &= y(t) \end{aligned}$$

for $a \le t \le b$.

1) Express $\dfrac{dy}{dx}$ in terms of derivatives with respect to t. Answer: $\dfrac{dy}{dx} = \boxed{}$

2) The tangent line to $\mathcal{C}$ when $t = t_0$ is $y = mx + b$ where m is $\boxed{}$ evaluated at $t = t_0$.

3) Express $\dfrac{d^2y}{dx^2}$ using derivatives with respect to t. Answer: $\dfrac{d^2y}{dx^2} = \boxed{}$

4) The arc length of $\mathcal{C}$, expressed as on integral with respect to t, is

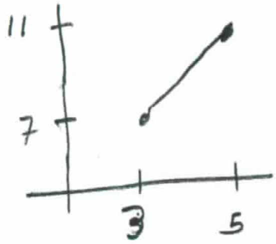
Arc Length = $\boxed{}$

Ex 1 The puffo is walking along the line $y = 2x+1$

1a Graph $y = 2x+1$ for $3 \le x \le 5$

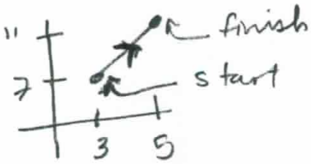
line thru pts $(3, 2(3)+1) = (3,7)$

need 2 pts & $(5, 2(5)+1) = (5,11)$



Now think of a puffo walking along this line segment.

1b. Param. this line segment w/ direction as indicated by arrow.



$x = t$ $\quad 3 \le t \le 5$

$y = 2t+1$

want $y = 2x+1$] $\Rightarrow y = 2(t)+1$
have: $x = t$

starts @ $t=3$, so @ $(3, 2(3)+1) = (3$

finishes @ $t=5$ so @ $(5, 2(5)+1) = ($

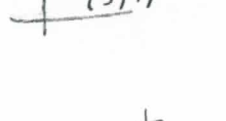
1c. Do 1b another way. <answers will vary> want $3 \le x \le 5$, $x = t^2$

$x = t^2$

$y = 2t^2+1$; $\sqrt{3} \le t \le \sqrt{5}$

need $y = 2x+1$

1d Param this line seg w/ puffo walken wcn pace but in the other direction, as indicated by the arrow



wrong $\cancel{x = t \quad 5 \le t \le 3}$
$\cancel{y = 2t+1}$

correct: take $x = -t$, $3 \le -t \le 5$, & "stay" on line

$\Updownarrow$ $\qquad\qquad \Updownarrow$

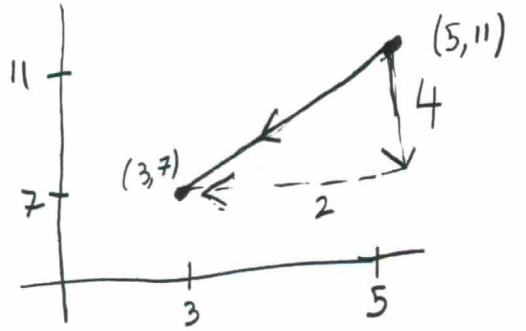
$-3 \ge t \ge 5$ $\qquad \boxed{\text{keep } y = 2x+1}$

$x = -t$ for $-5 \le t \le -3$

$y = -2t+1$

want $y = 2x+1 \xrightarrow{x=-t} 2(-t)+1$

1e do 1d another way



For $0 \le t \le 1$

$$x \overset{\text{or}}{=} x(t) = \boxed{5} + -2t \qquad \text{for } 0 \le t \le 1$$

$$y \overset{\text{or}}{=} y(t) = \boxed{11} + -4t$$

start ↑ as t varies .. move along line

Check:

switch

when $t=0$, start at $(x(0), y(0)) = (5-2(0), 11-4(0)) = (5,11)$

when $t=1$, end at $(x(1), y(1)) = (5-2(1), 11-4(1)) = (3,7)$

Equation of line is $y = 2x+1$:

$$2x(t)+1 \overset{A}{=} 2(5-2t)+1 \overset{A}{=} 11-4t = y(t)$$

If Niffy $\boxed{\dfrac{dy}{dx} \overset{?}{=} \dfrac{dy/dt}{dx/dt}}$

Well $\dfrac{dy}{dt} \underset{\text{Rule}}{\overset{\text{Chain}}{=}} \dfrac{dy}{dx} \cdot \dfrac{dx}{dt}$

in 1e $dy/dt = -4 \le 0$ ← why?

$dx/dt = -2 \le 0$

$dy/dx = -4/-2 = 2 \ge 0$

▶ Have a parameterized curve $\begin{bmatrix} x = x(t) \\ y = y(t) \end{bmatrix}$ for $0 \le t \le b$.

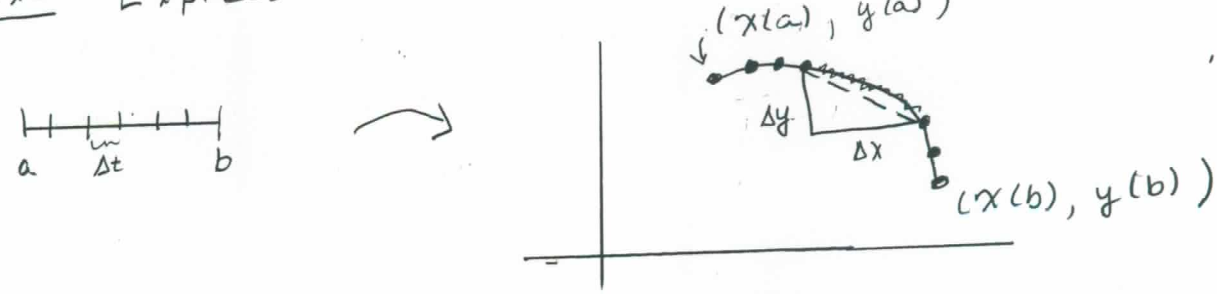
<u>Ex2</u> Find a general formula for $\frac{d^2y}{dx^2}$.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \overset{\text{i.e.}}{=} \frac{d\left(\frac{dy}{dx}\right)}{dx} \underset{(1)}{\overset{CR}{=}} \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

So
$$\boxed{\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}}$$

← ps - the way the book express this often confuses students, so let use this way.

<u>Ex3</u> Express the curve's length as an integral w.r.t. t.



Typical element true arc length

= the ~~curvy~~ part

≈ the hypothis dotted line

$= \sqrt{(\Delta x)^2 + (\Delta y)^2}$ ... think "Riemann sums"

$\overset{(A)}{=} \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}\ \Delta t$

Add all the typical element arc-length together and and then take $\lim_{\Delta t \to 0}$ to get

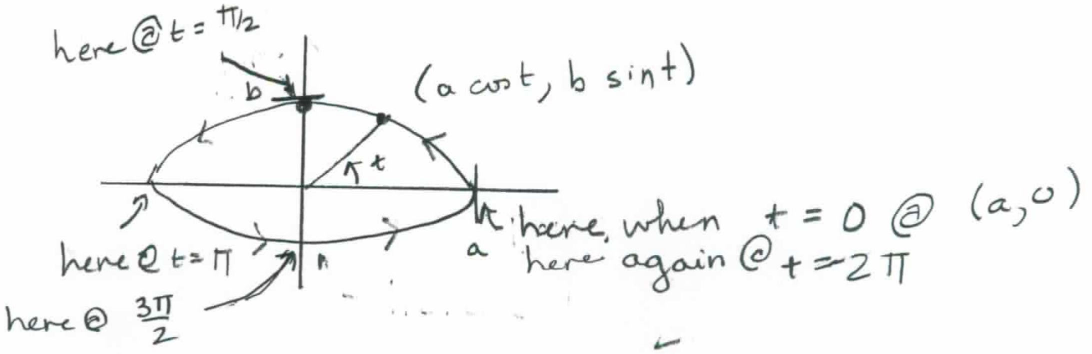
$$\boxed{\text{Arc length} = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt.}$$

Ex 4 Given $a>0$, $b>0$ and parameterization:

$$x = a\cos t \qquad \text{for} \quad 0 \le t \le 2\pi.$$
$$y = b\sin t$$

4a Sketch the curve & the parameterization traces out, indicating direction.

Well $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ ← ellipse



4b Find the equation of the tangent line to the curve when $t = \pi/4$. Express in form $y = mx + b$.

Point of tangency is $\left(a\cos\frac{\pi}{4},\ b\sin\frac{\pi}{4}\right) = \left(\frac{a\sqrt{2}}{2},\ \frac{b\sqrt{2}}{2}\right)$.

Slope of tangent line $= \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}}$ will need in 4c;

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{b\cos t}{-a\sin t} = \boxed{-\frac{b}{a}\cot t = \frac{dy}{dx}}$$

$$\left.\frac{dy}{dx}\right|_{\frac{\pi}{4}} = \frac{b\cos\pi/4}{-a\sin\pi/4} = \frac{-b}{a}\,\frac{\cos\pi/4}{\sin\pi/4} = \frac{-b}{a}.$$

So equation of tangent line (in pt.-slope form) is $\left(y - \frac{b\sqrt{2}}{2}\right) = \frac{-b}{a}\left(x - \frac{a\sqrt{2}}{2}\right)$.

$$\Rightarrow y = -\frac{b}{a}\left(x - \frac{a\sqrt{2}}{2}\right) + \frac{b\sqrt{2}}{2} = -\frac{b}{a}x + \frac{b\sqrt{2}}{2} + \frac{b\sqrt{2}}{2} = -\frac{b}{a}x + \sqrt{2}\,b$$

$$\boxed{y = -\frac{b}{a}x + \sqrt{2}\,b}$$

6

4.c Find $\frac{d^2y}{dx^2}$ @ $t = \pi/4$ (CR)

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \overset{\text{ie}}{=} \frac{d\left(dy/dx\right)}{dx} \overset{\text{CR}}{=} \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{d}{dx}\, x} = \frac{\frac{d}{dt}\left(-\frac{b}{a}\cot t\right)}{\frac{d}{dt}(a\cos t)}$$

$$= \frac{+\frac{b}{a}\csc^2 t}{-a\sin t} = \frac{-b\csc^3 t}{a^2}$$

$\csc \pi/4 = \frac{1}{\sin \pi/4} = \sqrt{2}$

$$\left.\frac{d^2y}{dx^2}\right|_{x=\pi/4} = \frac{-b(\sqrt{2})^3}{a^2} = \frac{-2\sqrt{2}\,b}{a^2}$$

4d. Express the length of the curve as an integral wrt t ⇓ dt

AL **pythogrean** $\displaystyle\int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$= \int_0^{2\pi} \sqrt{(-a\sin t)^2 + (b\cos t)^2}\, dt$$

$$\boxed{= \int_0^{2\pi} \sqrt{a^2\sin^2 t + b^2\cos t}\, dt}$$

or by symmetry

$= 4\cdot$ (AL in 1st Q.)

$$= \boxed{4\int_0^{\pi/2} \sqrt{a^2\sin^2 t + b^2\cos^2 t}\, dt}$$

1.5

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