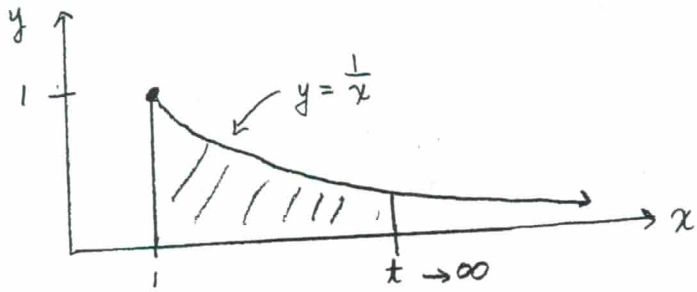
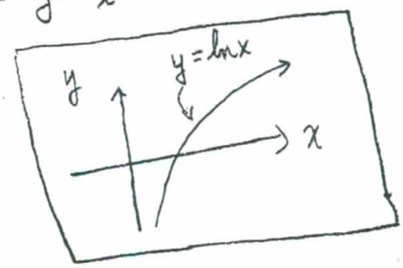
§7.8 Improper Integrals

(A useful handout. "Indeterminate Forms - L'Hôpital's Rule)

Ex 1 $\displaystyle\int_{x=1}^{x=\infty} \frac{dx}{x} \overset{\text{or}}{=} \int_{x=1}^{x=\infty} \frac{1}{x}\,dx =$ "area under curve $y=\frac{1}{x}$ from $x=1$ to $x=\infty$" $= ?$





$$\int_{x=1}^{x=\infty} \frac{dx}{x} = \lim_{t\to\infty}\left[\int_{x=1}^{x=t}\frac{dx}{x}\right] = \lim_{t\to\infty}\left[\ln|x|\,\Big|_{x=1}^{x=t}\right] = ?$$

$$= \lim_{t\to\infty}\left[\ln|t| - \ln 1\right] = \lim_{t\to\infty} \ln|t| = \infty.$$

So $\boxed{\displaystyle\int_{x=1}^{x=\infty}\frac{dx}{x} = \text{diverges}}$ or we could also say $\boxed{\displaystyle\int_{x=1}^{x=\infty}\frac{dx}{x} \text{ diverges to } \infty}$

Improper Integrals

1 DEFINITION OF AN IMPROPER INTEGRAL OF TYPE 1

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geqslant a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leqslant b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

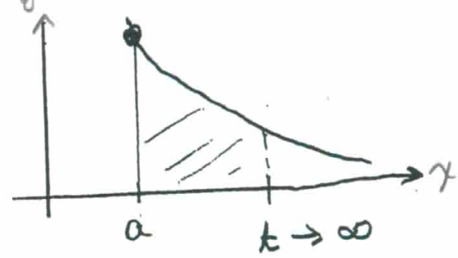
The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist. (DNE)

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

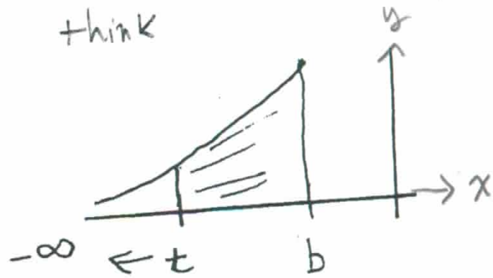
$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

In part (c) any real number a can be used (see Exercise 74).

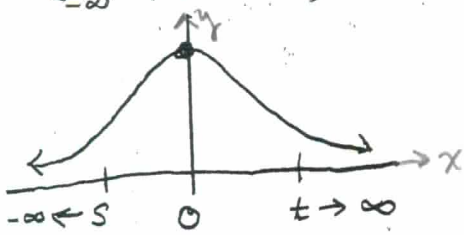
(a) For $\int_a^\infty f(x)\,dx$, think



(b) For $\int_{-\infty}^b f(x)\,dx$, think



(c) For $\int_{-\infty}^\infty f(x)\,dx$, think



$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx$$

$$= \lim_{s\to-\infty} \int_s^0 f(x)\,dx + \lim_{t\to\infty} \int_0^t f(x)\,dx$$

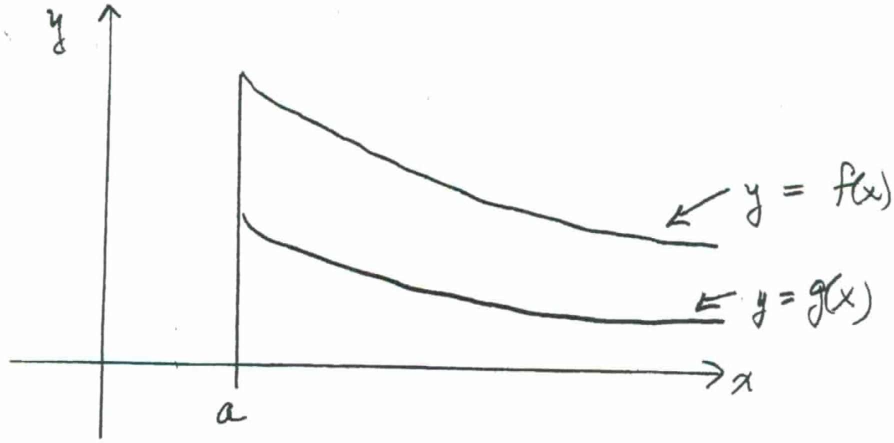
(*) If one, or both, of these limits DNE

then $\int_{-\infty}^\infty f(x)\,dx$ DNE.

COMPARISON THEOREM Suppose that f and g are continuous functions with $f(x) \geqslant g(x) \geqslant 0$ for $x \geqslant a$.

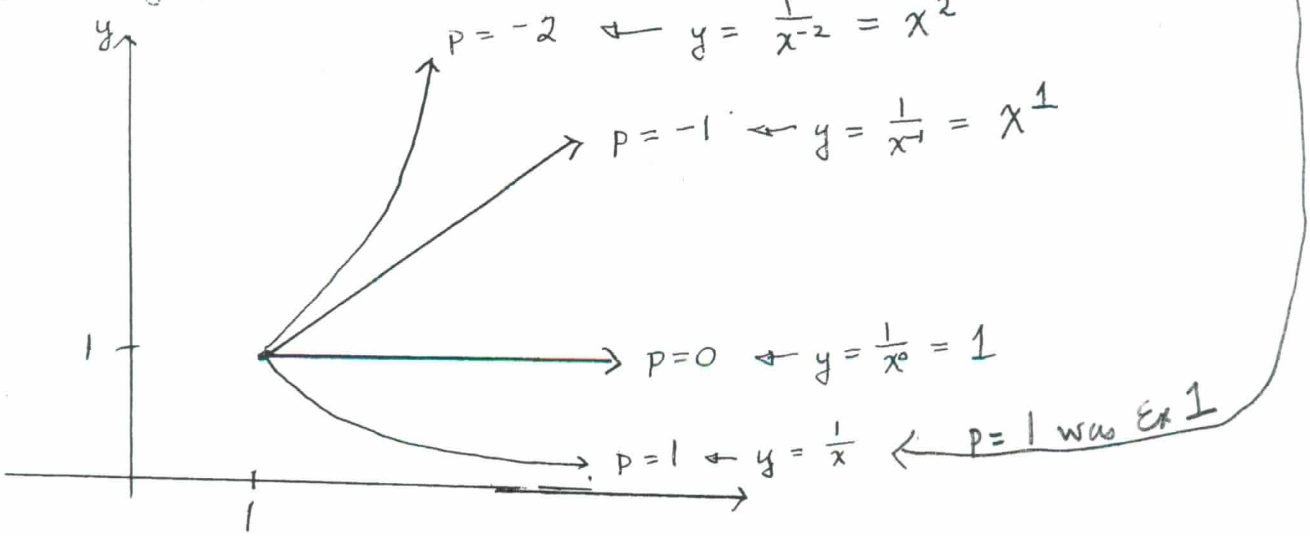
(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

(b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.



Ex 1 : beefed up. Consider the function $y = \frac{1}{x^p}$ for $x \geq 1$.

Let's graph this function, for varies values of $p \leq 1$.



Note If $p \leq 1$ and $x \geq 1$, then $\frac{1}{x^p} \geq \frac{1}{x}$.

So $\displaystyle\int_{x=1}^{x=\infty} \frac{1}{x^p}\,dx \geq \int_{x=1}^{x=\infty} \frac{1}{x}\,dx \overset{\text{Ex 1}}{=\!=} \infty$.

So $\displaystyle\int_{x=1}^{\infty} \frac{1}{x}\,dx$ diverges to ∞ if $p \leq 1$.

$\boxed{2}$ $\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p > 1$ and divergent if $p \leq 1$.

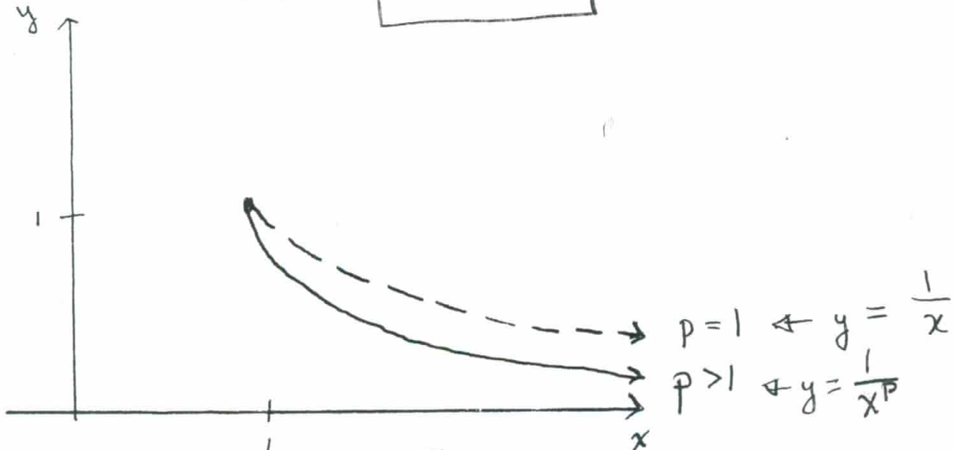
Already shown in Ex 1 - beefed up.
Let's do now in Ex 2.

← Will use in infinite series

very important.

Ex 2 Show that $\int_1^\infty \frac{dx}{x^p}$ converges if $p > 1$.

Let's first graph $\boxed{f(x) = \frac{1}{x^p}}$ for $x \geq 1$



$$\int_{x=1}^{x=\infty} \frac{1}{x^p}\,dx = \lim_{t\to\infty}\left[\int_{x=1}^{x=t} x^{-p}\,dx\right]$$

$$= \lim_{t\to\infty} \frac{x^{1-p}}{1-p}\Bigg|_{x=1}^{x=t}$$

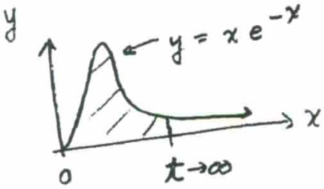
$$= \frac{1}{1-p}\lim_{t\to\infty}\left(t^{1-p} - 1\right)$$

Have $1 < p$ so $1-p < 0$

$$= \frac{1}{1-p}(0-1) = \frac{1}{p-1}$$

Ex 3. $\displaystyle\int_{x=0}^{x=\infty} x e^{-x}\,dx = ?$

Graph



Note to integrate $\int x e^{-x}\,dx$, one uses ... integration by parts.

So to integrate $\displaystyle\int_{x=0}^{x=\infty} x e^{-x}\,dx$, it is easer to first find $\int x e^{-x}\,dx$ and then evaluate the indefinite integral at the limits of integration ... why?

$$\int x e^{-x}\,dx = -x e^{-x} + \int e^{-x}\,dx = -x e^{-x} - e^{-x} + C$$

$$\boxed{\begin{array}{ll} u = x & dv = e^{-x}\,dx \\ du = dx & v = -e^{-x} \end{array}}$$

$$= -e^{-x}(x+1) + C$$

$$= \frac{x+1}{-e^{x}} + C.$$

I hate negative signs

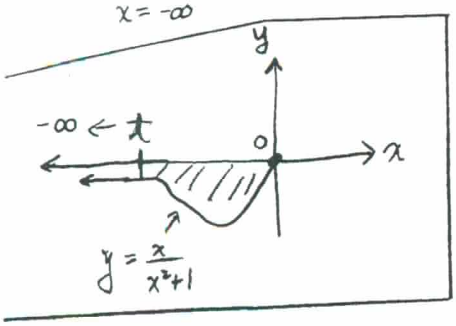
So
$$\int_{x=0}^{x=\infty} x e^{-x}\,dx = \lim_{t\to\infty}\int_{x=0}^{x=t} x e^{-x}\,dx = \lim_{t\to\infty} \frac{x+1}{-e^{x}}\Bigg|_{x=0}^{x=t} = \lim_{t\to\infty} \frac{x+1}{e^{x}}\Bigg|_{x=t}^{x=0}$$

$$= \lim_{t\to\infty}\left[\frac{1}{e^{0}} - \frac{t+1}{e^{t}}\right] = 1 - \left[\lim_{t\to\infty}\frac{t+1}{e^{t}}\right] \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}}$$

diff. wrt what variable

$$= 1 - \lim_{t\to\infty}\frac{1+0}{e^{t}} = 1 - \lim_{t\to\infty}\frac{1}{e^{t}} = 1 - 0 = \boxed{1}$$

Ex 4 $$\int_{x=-\infty}^{x=0} \frac{x}{x^2+1}\,dx = \lim_{t\to-\infty}\int_{x=t}^{x=0}\frac{x\,dx}{x^2+1} = \lim_{t\to-\infty}\frac{1}{2}\int_{x=t}^{x=0}\frac{2x\,dx}{x^2+1}$$



$$= \lim_{t\to-\infty}\frac{1}{2}\ln|x^2+1|\Bigg|_{x=t}^{x=0}$$

$$= \lim_{t\to-\infty}\Big[\frac{1}{2}\underbrace{\ln 1}_{0} - \frac{1}{2}\ln|t^2+1|\Big]$$

$$= -\frac{1}{2}\lim_{t\to-\infty}\ln|t^2+1| = -\frac{1}{2}\cdot\infty = -\infty.$$

$|t^2+1| \to \infty$

$$\boxed{\text{So } \int_{x=-\infty}^{x=0}\frac{x}{x^2+1}\,dx \text{ diverges to } -\infty}$$

Ex 5

$$\int_{x=-\infty}^{x=\infty} \frac{x\,dx}{x^2+1} = \int_{x=-\infty}^{x=17} \frac{x\,dx}{x^2+1} + \int_{x=17}^{x=\infty} \frac{x\,dx}{x^2+1}$$

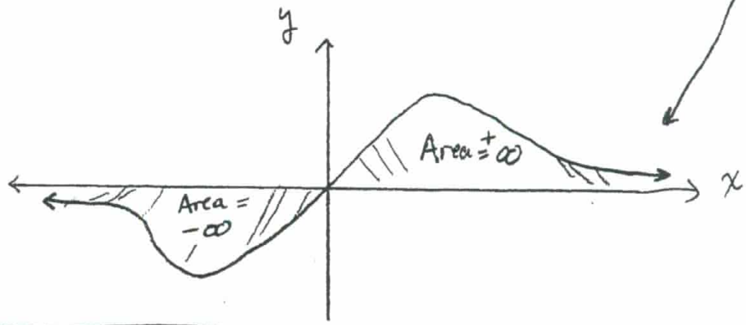
$$\overset{\text{or}}{\underset{\text{easier}}{=}} \underbrace{\int_{x=-\infty}^{x=0} \frac{x\,dx}{x^2+1}}_{\substack{\| \\ -\infty}} + \underbrace{\int_{x=0}^{x=\infty} \frac{x\,dx}{x^2+1}}_{\substack{\| \\ \infty}}$$

symmetry

$$f(x) = \frac{x}{x^2+1}$$

$$\Downarrow$$

$$f(-x) = -f(x)$$



$$\Rightarrow \boxed{\int_{x=-\infty}^{x=\infty} \frac{x\,dx}{x^2+1} \text{ diverges}} \leftarrow$$

but it does $\underline{\underline{\text{not}}}$ diverge to ∞

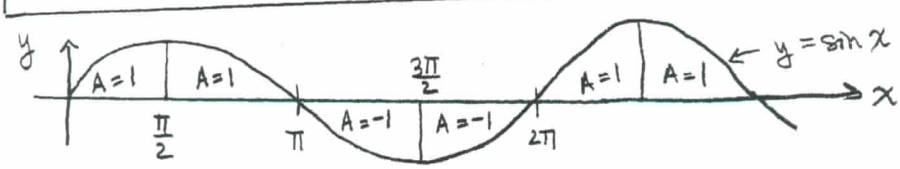
it does $\underline{\underline{\text{not}}}$ diverge to $-\infty$

Ex 6

$$\int_{x=0}^{x=\infty} \sin x\,dx = \lim_{t\to\infty} \int_{x=0}^{x=t} \sin x\,dx = \lim_{t\to\infty} (-\cos x)\Big|_{x=0}^{x=t} = \lim_{t\to\infty} \cos x \Big|_{x=t}^{x=0}$$

$$= \lim_{t\to\infty} [\cos 0 - \cos t] = 1 - \Big[\underbrace{\lim_{t\to\infty} \cos t}_{\text{oscillates between } -1 \text{ and } 1}\Big] = \text{DNE, oscillates btw 2 and}$$

$$\Rightarrow \boxed{\int_{x=0}^{x=\infty} \sin x\,dx \text{ diverges (because it oscillates between 0 and 2)}}$$



[3] DEFINITION OF AN IMPROPER INTEGRAL OF TYPE 2

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

The improper integral $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

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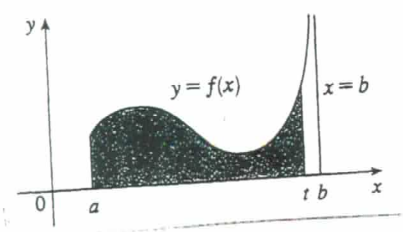
[3] For $\boxed{y = f(x)}$ that are continuous on $[a, b]$ except possibly :

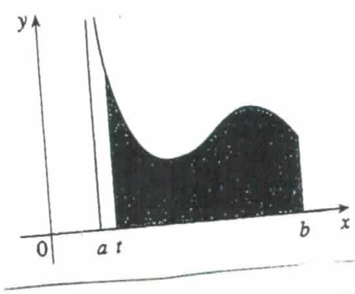
(a)

at $x = b$.



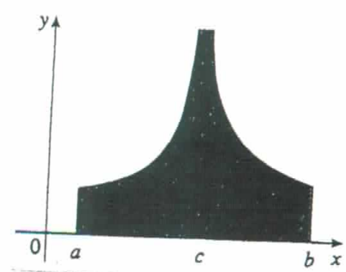
(b)

at $x = a$.

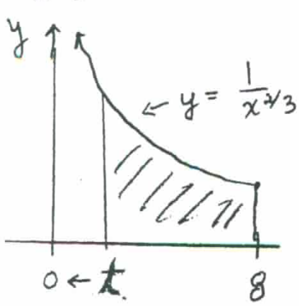


(c)

at $x = c$.
where $a < c < b$



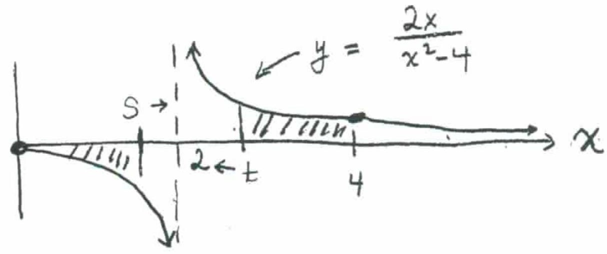
Ex 7. $\displaystyle\int_{x=0}^{x=8} \frac{dx}{x^{2/3}} = \lim_{t \to 0^+} \int_{x=t}^{x=8} x^{-2/3}\,dx = \lim_{t \to 0^+} 3x^{1/3} \Big|_{x=t}^{x=8=2^3}$



$$= \lim_{t \to 0^+} \left[3 \cdot (2^3)^{1/3} - 3t^{1/3}\right] = 3 \cdot 2 - 3 \lim_{t \to 0^+} t^{1/3}$$

$$= 6 - 3 \cdot (0) = \boxed{6}$$

To help with Example 8, let's first make a rough sketch of the graph of $f(x) = \frac{2x}{x^2-4}$ for $x \geq 0$.
The domain of $y = f(x)$ is $[0,\infty) \setminus \{2\} = [0,2) \cup (2,\infty)$.
$\lim_{x\to 2^+} \frac{2x}{x^2-4} = ?$ and $\lim_{x\to 2^-} \frac{2x}{x^2-4} = ?$
Next you can do the 1st Derivative Test to see when f is increasing and decreasing.
Then you can do the 2st Derivative Test to see when f is CCU and CCD.
If you need to review your Calc I for graphing - do it!



Ex 8. $\int_{x=0}^{x=4} \frac{2x\,dx}{x^2-4} = ?$

$$\int_{x=0}^{x=4} \frac{2x\,dx}{x^2-4} = \left[\lim_{s\to 2^-} \int_{x=0}^{x=s} \frac{2x\,dx}{x^2-4}\right] + \left[\lim_{t\to 2^+} \int_{x=t}^{x=4} \frac{2x\,dx}{x^2-4}\right]$$
$$= \left[\lim_{s\to 2^-} \ln|x^2-4| \,\Big|_{x=0}^{x=s}\right] + \left[\lim_{t\to 2^+} \ln|x^2-4| \,\Big|_{x=t}^{x=4}\right]$$
$$= \underbrace{\left[\lim_{s\to 2^-} \left(\ln|s^2-4| - \ln 4\right)\right]}_{s\to 2^- \;\Rightarrow\; |s^2-4|\to 0^+ \;\Rightarrow\; \ln|s^2-4|\to -\infty} + \underbrace{\left[\lim_{t\to 2^+} \left(\ln 12 - \ln|t^2-4|\right)\right]}_{t\to 2^+ \;\Rightarrow\; |t^2-4|\to 0^+ \;\Rightarrow\; \ln|t^2-4|\to -\infty}$$
$$= [-\infty] + [+\infty] \,. \quad \Leftarrow \text{THIS DOES NOT MAKE SENSE !}$$

So $\int_{x=0}^{x=4} \frac{2x\,dx}{x^2-4}$ diverges (or can also say DNE).

Ex 8. Revisited What is wrong with this way?

$$\int_{x=0}^{x=4} \frac{2x\,dx}{x^2-4} = \ln|x^2-4| \,\Big|_{x=0}^{x=4} = \ln 12 - \ln 4 = \ln\frac{12}{4} = \ln 3\,.$$

A common mistake is to NOT recognize an improper integral when you see him and then just (incorrectly) *blindly* integrate.

Ex 8. ReRevisited

$$\int_{x=0}^{x=\infty} \frac{2x\,dx}{x^2-4} = \int_{x=0}^{x=2} \frac{2x\,dx}{x^2-4} + \int_{x=2}^{x=17} \frac{2x\,dx}{x^2-4} + \int_{x=17}^{x=\infty} \frac{2x\,dx}{x^2-4}$$
$$= \underbrace{\left[\lim_{s\to 2^-} \int_{x=0}^{x=s} \frac{2x\,dx}{x^2-4}\right]}_{-\infty} + \underbrace{\left[\lim_{t\to 2^+} \int_{x=t}^{x=17} \frac{2x\,dx}{x^2-4}\right]}_{\infty} + \underbrace{\left[\lim_{u\to\infty} \int_{x=17}^{x=u} \frac{2x\,dx}{x^2-4}\right]}_{\text{who cares}}$$

So $\int_{x=0}^{x=\infty} \frac{2x\,dx}{x^2-4}$ diverges (or can also say DNE).