# Ch 6. Applications of the Definite Integral



## § 6.1 Area between (btw) 2 curves (this section was covered in Calc I.)

Recall from Calculus I the following ideas.

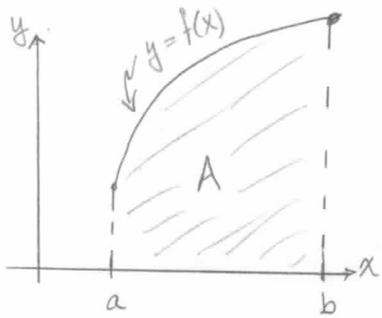
Problem: Find the $\boxed{\text{area}}$ of the region A that is bounded by:

- $x = a$ and $x = b$ (where $a < b$) $\Rightarrow$ so $a \le x \le b$
- a positive-valued (so it's graph is above the $x$-axis) continuous function
  $y = f(x)$ $\xrightarrow{\text{so}}$ $f : [a,b] \to [0,\infty)$

and

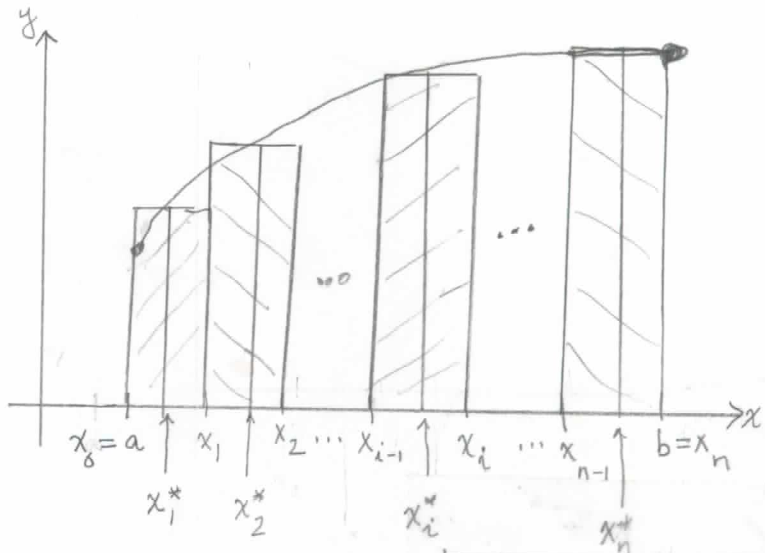
- the $x$-axis $\xrightarrow{\text{forshadowing}}$ $[x\text{-axis}] \Leftrightarrow [y = 0] \Leftrightarrow [y = g(x) \text{ w/ } g \equiv 0]$

Picture




$\boxed{\text{WANT}}$

Key idea: Approx. & take limit




Key idea: Riemann Sums

① Area $\approx \sum$ (area of typical rectangle)

$\Delta x$ $\qquad$ $\Delta x = \frac{b-a}{n}$

② Partition the interval $[a,b]$ along the $\boxed{x\text{-axis}}$ into $\boxed{n \text{ pieces}}$

③ take a selection $x_i^*$ from $[x_{i-1}, x_i]$

④ area of (a typical element)
= area of (a rectangle) = (height)(base) = $f(x_i^*)(\Delta x)$

⑤ form the corresponding Riemann Sum.

Area $\approx f(x_1^*)(\Delta x) + f(x_2^*)\Delta x + \dots + f(x_i^*)\Delta x + \dots + f(x_n^*)\Delta x$

$= \sum_{i=1}^{n} f(x_i^*)\Delta x$

⑥ Take the limit as $\Delta x \to 0$ (i.e. $n \to \infty$) to get $A = \int_a^b f(x)dx$

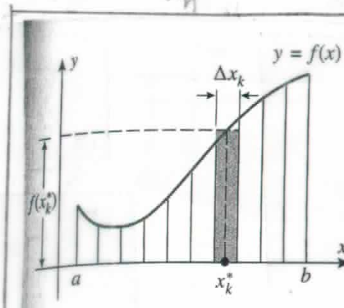



Figure 7.1.1

$$\sum_{k=1}^{n} f(x_k^*)\ \Delta x_k$$

Key idea $\qquad$ important

$$\int_a^b f(x)\ dx$$

Effect of the limit process on the Riemann sum.

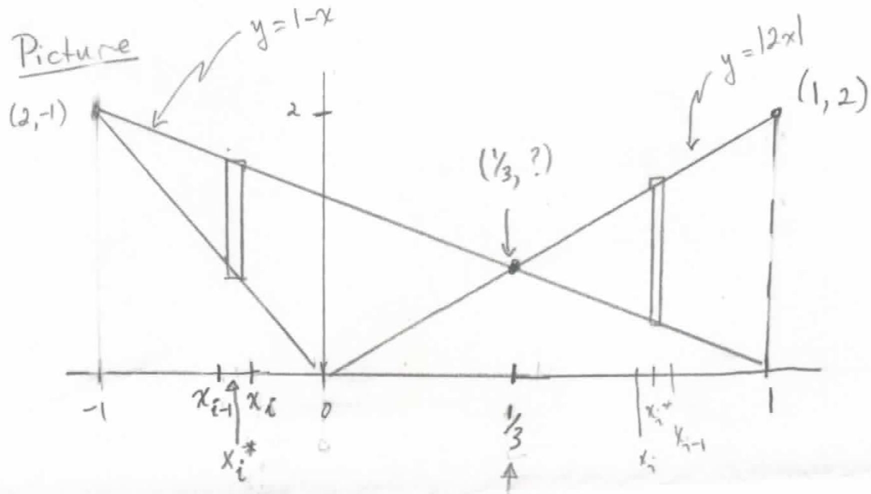
Figure 7.1.2

Calculus I Next we, in Calc I, beefed this idea up to finding the area between 2 curves.

Ex 1 Find the area A of the region enclosed by

$y = |2x|$ and $y = 1 - x$ between $x = -1$ and $x = 1$.

Picture




$2x = 1 - x \iff 3x = 1$

$|2x| = \begin{cases} -2x & x \le 0 \\ 2x & x \ge 0 \end{cases}$

Key idea

(1) partition $[-1, 1]$ along $x$-axis into $n$ pieces
$\Downarrow$
$\Delta x$

(2) make a selection $x_i^*$ from $[x_{i-1}, x_i]$

(3) form the corresponding Riemann Sums

Area $\approx$ sum of all those $\underbrace{\text{Riemann Rectangles}}_{\text{typical element}}$

Area of typical element $\approx$ (height) (length of base)

$\approx f(x_i^*) \, \Delta x$

total area $\approx \sum_{i=1}^{n} f(x_i^*) \, \Delta x$

(4) take limit as $\Delta x \to 0$

area $= \int_a^{x=b} f(x) \, dx$

Sol'n Area $\approx \sum$ (area of typical rectangle)

area of "typical element"

= area of "typical rectangle"

= (height) (base)

$$= \begin{cases} \underbrace{[(1-x) - |2x|]}_{\text{height}} \, \underbrace{\Delta x}_{\text{base}} & \text{if } -1 \le x \le \frac{1}{3} \\ [\,|2x| - (1-x)\,] \, \Delta x & \text{if } \frac{1}{3} \le x \le 1 . \end{cases}$$

So .... taking the limit....

Area = [Area for $-1 \le x \le \frac{1}{3}$] + [Area for $\frac{1}{3} \le x \le 1$]

$$= \int_{x=-1}^{x=1/3} [\underbrace{(1-x)}_{\text{top}} - \underbrace{|2x|}_{\text{bottom}}]\, dx + \int_{x=1/3}^{x=1} [\underbrace{|2x|}_{\text{top}} - \underbrace{(1-x)}_{\text{bottom}}]\, dx$$

$$= \int_{x=-1}^{x=0} [(1-x) - (-2x)]\, dx + \int_{x=0}^{x=1/3} [(1-x) - (2x)]\, dx + \int_{x=1/3}^{x=1} [(2x) - (1-x)]\, dx$$

$$= \int_{x=-1}^{x=0} (1+x)\, dx + \int_{x=0}^{x=1/3} (1-3x)\, dx + \int_{x=1/3}^{x=1} (3x-1)\, dx$$

© $\frac{1}{2} + \frac{1}{6} + \frac{2}{3} = \boxed{\frac{4}{3}}$

Oops... I just did an example before stating the theory.

Here is the theory,

## AREA BETWEEN $y = f(x)$ AND $y = g(x)$

We will now consider the following extension of the area problem.

> [redacted] **FIRST AREA PROBLEM.** Suppose that $f$ and $g$ are continuous functions on an interval $[a, b]$ and
>
> $$f(x) \geq g(x) \quad \text{for} \quad a \leq x \leq b$$
>
> [This means that the curve $y = f(x)$ lies above the curve $y = g(x)$ and that the two can touch but not cross.] Find the area $A$ of the region bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by the lines $x = a$ and $x = b$ (Figure 7.1.3$a$).

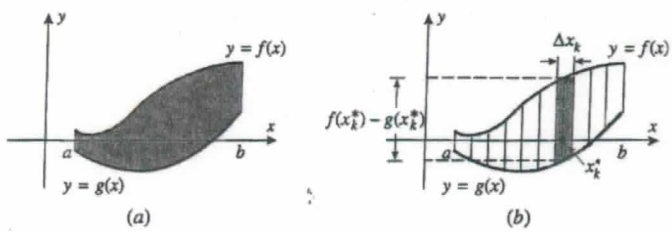
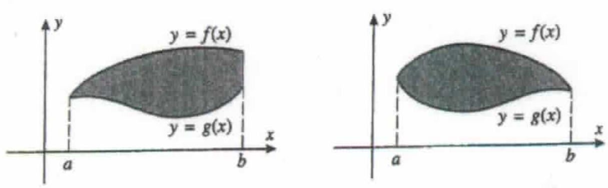



**Figure 7.1.3**

> [redacted] **AREA FORMULA.** If $f$ and $g$ are continuous functions on the interval $[a, b]$, and if $f(x) \geq g(x)$ for all $x$ in $[a, b]$, then the area of the region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$ is
>
> $$A = \int_a^b [f(x) - g(x)]\, dx \qquad (1)$$




The left-hand boundary reduces to a point.

Both side boundaries reduce to points.

Note

$y = \frac{x}{2} \Leftrightarrow x = 2y$

$y^2 = 8 - x \Leftrightarrow x = 8 - y^2$

Ex 2. Find the area $A$ of the region enclosed by

$$y = \frac{x}{2} \quad \text{and} \quad y^2 = 8 - x.$$

Oh no they do not give use " between $x = ?$ & $x = ?$ "

so got to find these!

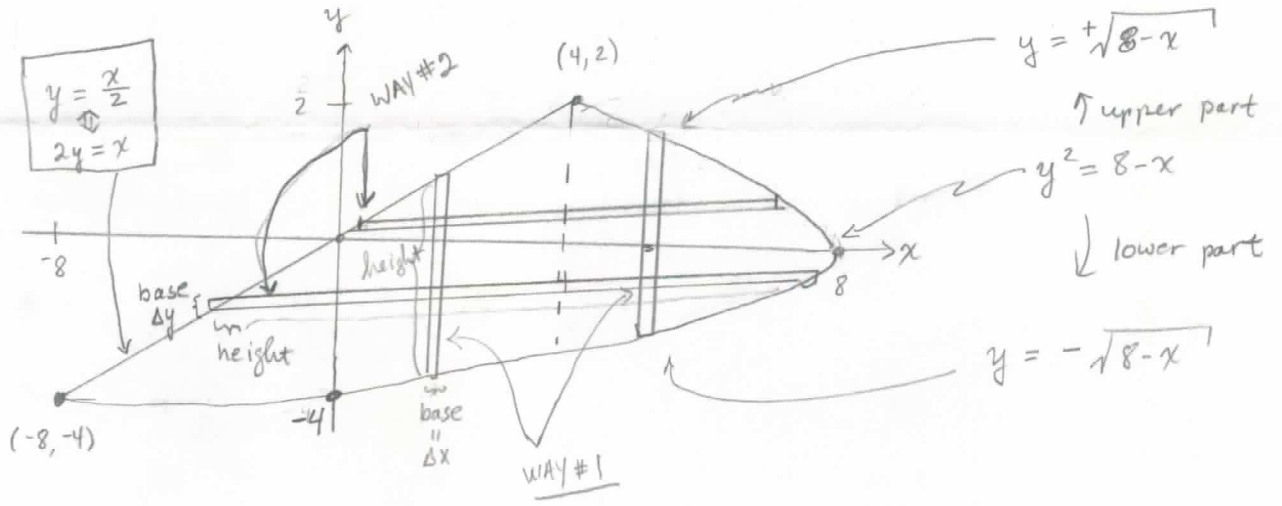
Points of intersection $\quad \boxed{x =} \quad 2y = 8 - y^2 \Rightarrow y^2 + 2y - 8 = 0 \Rightarrow (y-2)(y+4) = 0$

$\Rightarrow y = 2, -4$

$\Downarrow$

$(\underset{x=2y}{4}, \underset{y}{2})$ and $(\underset{x=2y}{-8}, \underset{y}{-4})$

Picture




Idea Want Area

So approx. area with rectangles

Form "typical element" = rectangles

Way # 1 Partition $x$-axis $\Rightarrow \Delta x,\ dx$ $\Rightarrow$ $\boxed{\text{everything in terms of } x}$

Know area of typical element = (height) (base)

$$= \begin{cases} \left[\frac{x}{2} - (-\sqrt{8-x})\right] \Delta x & -8 \le x \le 4 \\ \left[\underbrace{\sqrt{8-x}} - (-\sqrt{8-x})\right] \Delta x & 4 \le x \le 8 \end{cases}$$

$$\left[\begin{pmatrix}\text{height}\\\text{top}\end{pmatrix} - \begin{pmatrix}\text{height}\\\text{bottom}\end{pmatrix}\right] \underset{\text{base}}{\uparrow}$$

So / limit

$$\text{Area} = \int_{x=-8}^{x=4} \left[\frac{x}{2} - (-\sqrt{8-x})\right] dx + \int_{x=4}^{x=8} \left[\sqrt{8-x} - (-\sqrt{8-x})\right] dx$$

$$= \int_{x=-8}^{x=4} \left[\frac{x}{2} + \underset{\substack{\uparrow \\ u = 8-x}}{(8-x)^{1/2}}\right] dx + \int_{x=4}^{x=8} 2\,(8-x)^{1/2}\, dx$$

$\overset{c}{=}$ 36.

Way # 2 Partition $y$-axis $\Rightarrow \Delta y,\ dy$ $\Rightarrow$ $\boxed{\text{everything in terms of } y}$

Know area of typical element = (height) (base)

$$= \left[\underbrace{(8-y^2)}_{\substack{\text{height}\\\text{top}}} - \underbrace{(2y)}_{\substack{\text{height}\\\text{bottom}}}\right] \underbrace{\Delta y}_{\text{base}}$$

So / limit

$$\text{Area} = \int_{y=-4}^{y=2} \left[(8-y^2) - (2y)\right] dy$$

$$= \int_{y=4}^{2} \left(-y^2 - 2y + 8\right) dy \overset{c}{=} 36.$$

<u>Oopps</u> - I just did another example before giving the theory,
Here is the theory,

## REVERSING THE ROLES OF $x$ AND $y$

Sometimes it is possible to avoid splitting a region into parts by integrating with respect to $y$ rather than $x$. We will now show how this can be done.

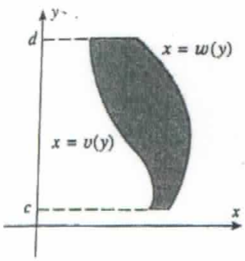


Figure 7.1.11

> **SECOND AREA PROBLEM.** Suppose that $w$ and $v$ are continuous functions of $y$ on an interval $[c, d]$ and that
>
> $$w(y) \geq v(y) \quad \text{for} \quad c \leq y \leq d$$
>
> [This means that the curve $x = w(y)$ lies to the right of the curve $x = v(y)$ and that the two can touch but not cross.] Find the area $A$ of the region bounded on the left by $x = v(y)$, on the right by $x = w(y)$, and above and below by the lines $y = d$ and $y = c$ (Figure 7.1.11).

Proceeding as in the derivation of (1), but with the roles of $x$ and $y$ reversed, leads to the following analog of 7.1.2.

> **AREA FORMULA.** If $w$ and $v$ are continuous functions and if $w(y) \geq v(y)$ for all $y$ in $[c, d]$, then the area of the region bounded on the left by $x = v(y)$, on the right by $x = w(y)$, below by $y = c$, and above by $y = d$ is
>
> $$A = \int_c^d [w(y) - v(y)]\, dy \qquad (4)$$

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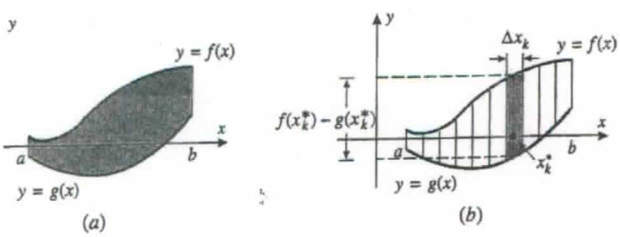
$\boxed{\text{Recap}}$

If partition $x$-axis $\Rightarrow$ everything in terms of $x$



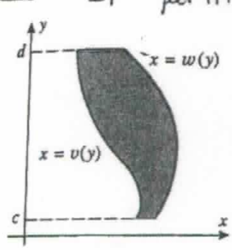

area of typical rectangle

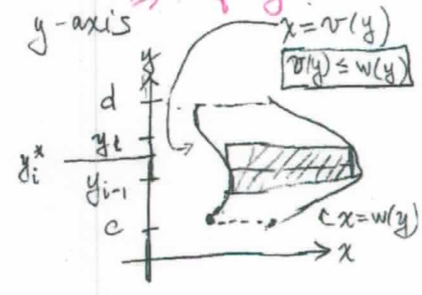
$= \text{(height)} \quad \text{(base)}$

$= [f(x_k^*) - g(x_k^*)] \quad (\Delta x)$

$$\text{Area} = \int_a^b [f(x) - g(x)]\, dx$$

If partition $y$-axis $\Rightarrow$ everything in terms of $y$.




area of typical rectangle

$= \text{(height)} \quad \text{(base)}$

$= [w(y_i^*) - v(y_i^*)] \quad (\Delta y)$

$$\text{Area} = \int_c^d [w(y) - v(y)]\, dy$$

6.7