§ 7.1 Integration by Parts

Remark Integration by Parts "undoes" the product rule so it is used lots.

Derive the Integration by Parts Formula

Have $y = u(x)$ and $y = v(x)$.

$\boxed{\text{Product Rule}}$ $D_x(uv) = u'v + uv'$

$\Downarrow$ integrate both sides and use F.T.C.

$$uv = \int (u'v + uv')\,dx$$

$$= \int v(x)\,\underbrace{u'(x)\,dx}_{\Downarrow} + \int u(x)\,\underbrace{v'(x)\,dx}_{\Downarrow}$$

$$= \int v\,du + \underbrace{\int u\,dv}_{\text{now solve for him to get}}$$

$\boxed{\text{Parts}}$ $\boxed{\int u\,dv = uv - \int v\,du}$

(*) $u = \quad \leftrightarrow \quad dv =$

$du = \quad \quad v =$

consistent with book so I'll do as such

Let's try to keep this up on the board for awhile

(*) If $dv = f(x)\,dx$, then $v = \int dv = \int f(x)\,dx$

Idea $\underbrace{\int u\,dv}$ is hard, $\underbrace{\text{so pick } u \,\&\, v}$ so that $\underbrace{\int v\,du}$ is easy.

$\downarrow$

How to do?

Well – lots of practice.

Homework: Look at the examples in the book.

You can think of "Parts Problems" as breaking up into

5 Lessons.

Let's write these 5 Lessons down now <on the side board>

and then we will do examples of each lesson.

Parts Problem Lessons.

Lesson 1 For $\int x^n f(x)\,dx$, try $u = x^n$ & $dv = f(x)\,dx$,

$$\Downarrow$$

This often reduces x^n to x^{n-1}.

$$V = \underbrace{\int f(x)\,dx}$$

Will need to be able to find V.

Lesson 2 Creatively look for a dv that is easy to integrate.

For then, $V = \int dv$.

Lesson 3 Bring to the other side idea.

Lesson 4 For $\int f(x)\,dx$, if the integrand $y = f(x)$ is easy to differentiate but hard to integrate, then try letting

$u = f(x)$ and $dV = dx$.

$\Downarrow$ note

$du = f'(x)\,dx \leftarrow$ can do.

Lesson 5 None of the above

Examples of Lesson 1

Ex 1a

$$\int x e^x \, dx = x e^x - \int e^x \, dx = \boxed{x e^x - e^x + C}$$

$$\boxed{\begin{array}{ll} u = x & \leftrightarrow dv = e^x \, dx \\ du = dx & v = e^x \end{array}}$$

Ex 1b

$$\int x^2 e^x \, dx = x^2 e^x - 2\int x e^x \, dx = x^2 e^x - 2\left(x e^x - \int e^x \, dx\right)$$

$$\boxed{\begin{array}{ll} u = x^2 & \leftrightarrow dv = e^x \, dx \\ du = 2x \, dx & v = e^x \end{array}}$$

↓ be careful, comon place for algebra error.

$$= x^2 e^x - 2x e^x + 2\int e^x \, dx$$

$$= \boxed{x^2 e^x - 2x e^x + 2e^x + C}$$

Note, there is a "d(variable)" on the LHS so there must be a "d(variable)" on the RHS.

Ex 1c How many times would we have to use Parts to find $\int x^{17} e^x \, dx$?

Answer : 17. Why?

Example of Lesson 2

Ex 2 $\int \sec^3 x \, dx$.

Well $\frac{d}{dx} \tan x = \sec^2 x \Rightarrow \tan x = \int \sec^2 x \, dx$.

So Lesson 2 suggests to try $dv = \sec^2 x \, dx$.

$\Downarrow$ note

$v = \tan x$

easy to find.

So we think of as

$$\int \sec^3 x \, dx = \int \underbrace{(\sec x)}_{u} \underbrace{(\sec^2 x \, dx)}_{dv}$$

continued →

Ex 2 continued

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec x \tan^2 x\,dx = \text{②}$$

Lesson 2

$u = \sec x \quad \longleftrightarrow \quad dv = \sec^2 x\,dx$

$du = \sec x \tan x\,dx \quad \quad v = \tan x$

Know $\cos^2 x + \sin^2 x = 1$ and $\tan x = \dfrac{\sin x}{\cos x}$

$$\Downarrow$$

$$\frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\Downarrow$$

$$1 + \tan^2 x = \sec^2 x$$

$$\begin{aligned}
\text{②} &= \sec x \tan x - \int \sec x(\sec^2 x - 1)\,dx \\
&= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx \\
&= \sec x \tan x - \int \sec^3 x\,dx + \ln|\sec x + \tan x| .
\end{aligned}$$

Let's step back and see what we have so far.

$$\boxed{\underbrace{\int \sec^3 x\,dx}_{\text{"a"}} = \underbrace{\sec x \tan x + \ln|\sec x + \tan x|}_{\text{"b"}} - \underbrace{\int \sec^3 x\,dx}_{\text{"a"}}}$$

How would you solve $a = b - a$? Well, "bring "a" to other side"

$$\begin{array}{rcl} a &=& b - a \\ +a & & +a \\ \hline 2a &=& b \end{array} \Rightarrow a = \frac{b}{2} .$$ Let's do it.

So bring $-\int \sec^3 x\,dx$ on the RHS to the LHS as $+\int \sec^3 x\,dx$:

$$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x|$$

$$\boxed{\int \sec^3 x\,dx = \frac{\sec x \tan x + \ln|\sec x + \tan x|}{2} + C}$$

Another Example using Lesson 3

Ex 3 $\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx = \text{↲}$

$$\boxed{\begin{array}{ll} u = \sin x & \leftrightarrow dv = e^x\,dx \\ du = \cos x\,dx & \rightleftarrows v = e^x \end{array}}$$

$$\boxed{\begin{array}{ll} u_2 = \cos x & \leftrightarrow dv_2 = e^x\,dx \\ du_2 = -\sin x\,dx & \rightleftarrows v_2 = e^x \end{array}}$$

$\hookrightarrow = e^x \sin x - \left[e^x \cos x - {}^{-}\!\int e^x \sin x\,dx \right] = e^x(\sin x - \cos x) - \int e^x \sin x\,dx.$

So we have :

$$\Rightarrow \int e^x \sin x\,dx = e^x(\sin x - \cos x) - \int e^x \sin x\,dx$$

$$+ \int e^x \sin x\,dx \qquad\qquad + \int e^x \sin x\,dx$$

"take to other side"

$$\Rightarrow 2\int e^x \sin x\,dx = e^x(\sin x - \cos x)$$

$$\Rightarrow \boxed{\int e^x \sin x\,dx = \frac{e^x(\sin x - \cos x)}{2} + C.}$$

Remark Example 3 is the book's Example 4, page 455. The book did it similarly, using "bring to other side idea" and Parts twice, but with

$$\boxed{\begin{array}{ll} u_1 = e^x & dv_1 = \sin x\,dx \\ du_1 = e^x\,dx & v_1 = -\cos x \end{array}} \qquad \boxed{\begin{array}{ll} u_2 = e^x & dv_2 = \cos x\,dx \\ du_2 = e^x\,dx & v_2 = \sin x \end{array}}.$$

This also works – take a look. What does not work is to once let $u = e^x$ and the other time $dv = e^x\,dx$. You will end up with $0 = 0$, which is true but not helpful.

Recall Lesson 4 — If integrand is easy to differentiate but hard to integrate, then try letting u = the integrand.

Ex 4a $\int_1^{e^2} \ln x \, dx$. Lesson 4 $\Rightarrow$

$$\boxed{\begin{array}{ll} u = \ln x & \longleftrightarrow \; dv = dx \\ du = \dfrac{dx}{x} & \longleftrightarrow \; v = x \end{array}}$$

Way #1

$$\int \ln x \, dx = x \ln x - \int x \frac{dx}{x} = x \ln x - \int dx = x \ln x - x + C$$

$$\Rightarrow \int_1^{e^2} \ln x \, dx = [x \ln x - x] \Big|_{x=1}^{x=e^2} = [e^2 \ln e^2 - e^2] - [1 \ln 1 - 1]$$

$$= [e^2(2) - e^2] - [0 - 1] = \boxed{e^2 + 1}$$

Way #2

$$\int_1^{e^2} \ln x \, dx \overset{\text{parts}}{=} x \ln x \Big|_{x=1}^{x=e^2} - \int_{x=1}^{x=e^2} dx$$

$$= [e^2 \ln e^2 - 1 \ln 1] - \left[x \Big|_{x=1}^{x=e^2}\right]$$

$$= [2e^2 - 0] - [e^2 - 1] = \boxed{e^2 + 1}.$$

A common mistake is to forget to evaluate the $u \cdot v$ at $\Big|_{x=a}^{x=b}$,

here $x \ln x \Big|_{x=1}^{x=e^2}$.

WRONG

$$\underbrace{\int_1^{e^2} \ln x \, dx}_{\text{This is a definite integral so this is a number}} = x \ln x - \int_1^{e^2} dx = \underbrace{x \ln x - [e^2 - 1]}_{\text{This is a function of } x.}$$

Ex 4b $\int \ln(x+17)\,dx$

WAY #1

$$u = \ln(x+17) \qquad dv = dx$$
$$du = \frac{dx}{x+17} \qquad v = x$$

$$\frac{x}{x+17} = \frac{x+17}{x+17} + \frac{-17}{x+17} = 1 + \frac{-17}{x+17}$$

or do long division

$$\begin{array}{r} 1 \\ x+17 \overline{)\; x} \\ -(x+17) \\ \hline -17 \end{array}$$

$$\int \ln(x+17)\,dx = x\ln(x+17) - \int \frac{x}{x+17}\,dx$$

$$= x\ln(x+17) - \int \left[1 + \frac{-17}{x+17}\right] dx$$

$$= x\ln(x+17) - \int dx + 17\int \frac{dx}{x+17}$$

$$= x\ln(x+17) - x + 17\ln(x+17) + c$$

$$= \boxed{(x+17)\ln(x+17) - x + c}$$

WAY #2

$$u = \ln(x+17) \qquad dv = dx$$
$$du = \frac{1}{x+17}\,dx \qquad v = x + \underline{\underline{17}}$$

$$\int \ln(x+17)\,dx = (x+17)\ln(x+17) - \int dx$$

$$= \boxed{(x+17)\ln(x+17) - x + c}$$

Ex 4c Derive a reduction formula for $\int (\ln x)^n\,dx$ for $n \geq 1$.

Recall $\ln(x^n) = n\ln x$ but $(\ln x)^n \neq n\ln x$.

Rmk $\ln(x^n) = n \ln x$

$(\ln x)^n \neq n \ln x$

Ex 4c Continued $\int (\ln x)^n \, dx = ?$

Lesson 4 $\Rightarrow$ $u = (\ln x)^n \longleftrightarrow dv = dx$

$du = n(\ln x)^{n-1} \cdot \frac{1}{x} \, dx \longleftarrow v = x$

$$\boxed{\int (\ln x)^n \, dx = x(\ln x)^n - n\int (\ln x)^{n-1} \, dx}$$

Reduce n to $n-1$

Ex 4d. $\int (\ln x)^2 \, dx \underset{n=2}{\overset{\text{Ex 4c}}{=}} x(\ln x)^2 - 2\int \ln x \, dx$

$\underset{n=1}{\overset{\text{Ex 4c}}{=}} x(\ln x)^2 - 2\left(x(\ln x)^1 - 1\int (\ln x)^0 \, dx\right)$

$= x(\ln x)^2 - 2x \ln x + 2\int dx$

$$= \boxed{x \ln^2 x - 2x \ln x + 2x + c}$$

Remark Know $(\sin x)^2 = \sin^2 x$

shorthand

likewise $(\ln x)^2 = \ln^2 x$

Lesson 5 None of the above lessons.

Wrap-up Let's look at book's §7.1 - Parts Examples.

2→ Look at our Lessons 1-5.

Ex 1. $\int x \sin x \, dx$ Lesson 1. $u = x$ $dv = \sin x \, dx$

Ex 2. $\int \ln x \, dx$ Lesson 4 $u = \ln x$ $dv = dx$

Ex 3. $\int t^2 e^t \, dt$ Lesson 1 $u_1 = t^2$ $dv = e^t dt$

do parts <u>2</u> times

Ex 4. $\int e^x \sin x \, dx$ Lesson 3 what's up w/ u-dv? <our Ex 3 today>

Ex 5. $\int \tan^{-1} x \, dx$ Lesson 4 $u = \tan^{-1} x$ $dv = dx$

$du = \dfrac{dx}{1+x^2}$ $v = x$

Ex 6 $\int \sin^n x \, dx$ Lesson 2 $u = \sin^{n-1} x$ $dv = \sin x \, dx$

$\downarrow$? $\downarrow$ easy to do

$du = (n-1)\sin^{n-2} x \, dx$ $V = -\cos x$

$\updownarrow$

Reduction formula