§ 11.3 Integral Test

(1) For § 11.3 – § 11.6, we want to know if a series:

☐ converges (abbreviated conv.)

or

☐ diverges (abbreviated divg.).

If it does converge, we do not worry what it's sum is, for such is a 500-level math problem.

(2) Let's get out the handout for § 11.3, titled "Integral Test", and read through and discuss the first page.

(3) As an overview of upcoming sections, let's get out the handout titled "Infinite Series - Summary".

Look at bottom half of first page. We will learn 6 tests:

(1) integral test (§ 11.3)

$\left.\begin{array}{l}\text{(2) Comparison test (CT)}\\ \text{(3) limit comparison test (LCT}\end{array}\right\}$ (§ 11.4)

$\left.\begin{array}{l}\text{(4) Ratio Test}\\ \text{(5) Root Test}\end{array}\right\}$ (§ 11.6)

(6) Alternating Series Test (AST) (§ 11.5).

(4) Let's get started on the integral test. Please take out the "Integral Test" handout and look at page 2.

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INTEGRAL TEST

- Let $\sum a_n$ be a positive-term series.
- Find a function $f(x)$ such that

  (1) $f(n) = a_n$ for $n \in \mathbb{N}$

  (2) f is decreasing for $x \geq 1$ (often check this by showing $f' < 0$)

  (3) f is continuous for $x \geq 1$

  (4) so $f(x) \geq 0$ for $x \geq 1$.

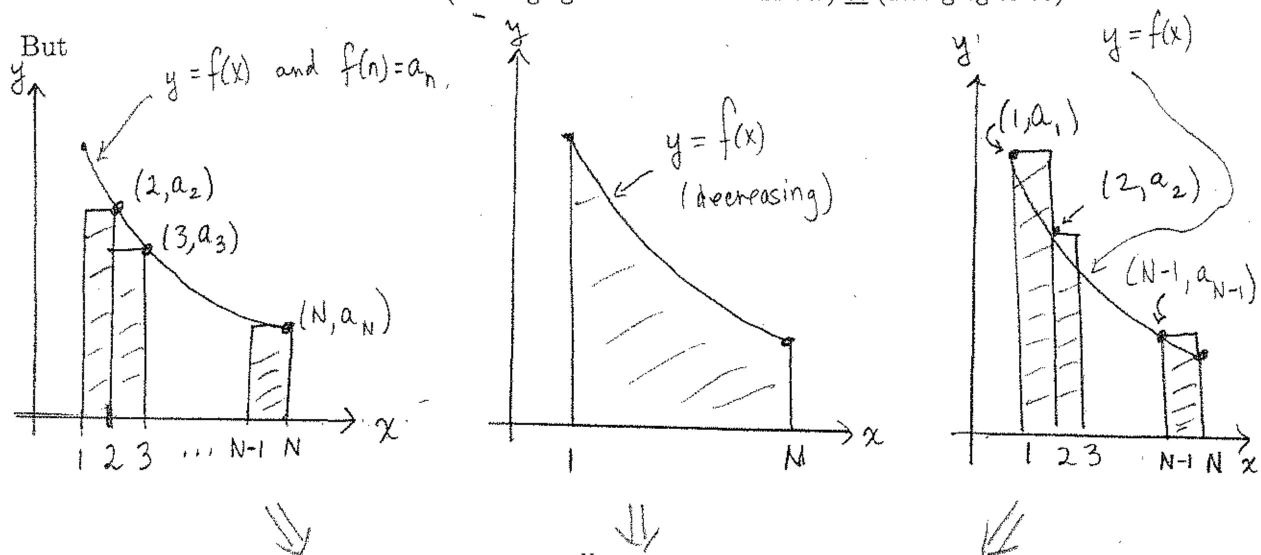
Then the series $\sum_{n=1}^{\infty} a_n$ and the improper integral $\int_1^{\infty} f(x)\,dx$ either:

  (1) both converge (to different numbers most likely)

  (2) both diverge .

This is because $\{\sum_{n=1}^{N} a_n\}_{N\in\mathbb{N}}$ and $\{\int_1^N f(x)\,dx\}_{N\in\mathbb{N}}$ are both non-decreasing sequences and so each has the choice of either (converging to some finite number) <u>or</u> (diverging to ∞).

But



$$a_2 + a_3 + \ldots + a_N \quad \leq \quad \int_1^N f(x)\,dx \quad \leq \quad a_1 + a_2 + \ldots + a_{N-1}$$

Now take the limit as $N \to \infty$ to see that

$$\sum_{n=2}^{\infty} a_n \quad \leq \quad \int_1^{\infty} f(x)\,dx \quad \leq \quad \sum_{n=1}^{\infty} a_n \; .$$

Ex 1 Very Important $\boxed{\text{p-series}}$

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^p = \sum_{n=1}^{\infty}\frac{1}{n^p} = 1+\left(\frac{1}{2}\right)^p+\left(\frac{1}{3}\right)^p+\left(\frac{1}{4}\right)^p+\dots = \begin{cases}\text{converges} & p>1\\ \text{diverges} & p\leqslant 1\end{cases}$$

If $p=1$, it's called the harmonic series. ← The harmonic series was discussed in §11.2

Proof of convergence and divergence of the p-series.

Case 1: $p=0$. $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^p = \sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^0 = \sum_{n=1}^{\infty} 1 = 1+1+1+1+\dots = \infty$.

So $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^p$ diverges (to ∞).

Case 2: $p<0$. So $-p>0$. So

$$\lim_{n\to\infty}\left(\frac{1}{n}\right)^p = \lim_{n\to\infty} n^{-p} \overset{-p>0}{=\!=} \infty \neq 0.$$

So $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^p$ diverges by the n^{th} term test for divergence.

Case 3: $p>0$. We will use the Integral Test so first we need to check the conditions of the Integral Test.

Here

$$a_n = \left(\frac{1}{n}\right)^p = \frac{1}{n^p} \quad \text{for } n=1,2,3,\dots$$

So let

$$f(x) = \frac{1}{x^p} = x^{-p} \quad \text{for } x \geq 1.$$

Let's check the 4 conditions of the integral test.

(1) $f(n)=a_n$ for all $n\in\mathbb{N}$? Yes because $f(n)=\frac{1}{n^p}=a_n$.

(2) f is decreasing for $x\geq 1$? Yes, by 1st derivative test since

$$f'(x) = -px^{-p-1} = \frac{-p}{x^{p+1}} < 0$$

$x>0$

↳ continued

(3) f is continuous for $x \geq 1$.? Yes, clear

(4) $f(x) \geq 0$ for $x \geq 1$? Yes, clear.

So the conditions of the Integral Test are satisfied so we can apply the integral test. The Integral Test says that

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{and} \quad \int_{x=1}^{x=\infty} \frac{1}{x^p}\,dx$$

"do the same thing", i.e.

- both converge (to a finite number)
- both diverge (to ∞).

From the lecture notes on Improper Integrals (§ 7.8),

$$\int_{x=1}^{x=\infty} \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ \text{diverges to } \infty & \text{if } p \leq 1. \end{cases}$$

So

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{converges (to some \#, don't know what)} & \text{if } p > 1 \\ \text{diverge (to } \infty) & \text{if } p \leq 1. \end{cases}$$

Ex 2 $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}$ ☐ converges ☐ diverges

Integral Test

Let $f(x) = \dfrac{\ln x}{x}$ for $x \geq \boxed{2}$.

Need to check conditions of the 4 conditions of the integral test:

(1) $f(n) = a_n$ for $n = \{2, 3, 4, \ldots\}$? Yes, by design

(2) f is decreasing for $x \geq \boxed{2}$? Let's use 1st derivative test.

$$f'(x) = \frac{\frac{1}{x}(x) - \ln x}{x^2} = \frac{1 - \ln x}{x^2} \overset{?}{<} 0$$

$$\Leftrightarrow 1 - \ln x < 0 \Leftrightarrow 1 < \ln x \Leftrightarrow \exp 1 < \exp \ln x \Leftrightarrow \overset{\substack{2.7\\55}}{e} < x$$

So f is decreasing for $\boxed{x \geq 3}$, but this is enough since "it doesn't matter where you start" Thm (§11.2)

(3) f is continuous for, well now, $x \geq 3$? Yes, clear

(4) $f(x) \geq 0$ for $x \geq 3$? Yes, clear.

Integral Test $\Rightarrow \displaystyle\sum_{n=3}^{\infty} \frac{\ln n}{n}$ and $\displaystyle\int_{x=3}^{x=\infty} \frac{\ln x}{x}\,dx$ "do the same thing"

$$\int_{x=3}^{x=\infty} \frac{\ln x}{x}\,dx = \lim_{b\to\infty} \int_{x=3}^{x=b} \frac{\ln x}{x}\,dx = \lim_{b\to\infty} \int_{u=\ln 3}^{u=\ln b} u\,du = \text{?}$$

$$\boxed{\begin{aligned} u &= \ln x \\ du &= \frac{dx}{x} \end{aligned}} \quad = \lim_{b\to\infty} \frac{u^2}{2} \Bigg|_{u=\ln 3}^{u=\ln b} = \frac{1}{2} \lim_{b\to\infty} \left[(\ln b)^2 - (\ln 3)^2\right].$$

$$= \frac{1}{2}\left[\left(\lim_{b\to\infty} (\ln b)^2\right) - (\ln 3)^2\right] = \infty.$$

So, by Integral Test, $\displaystyle\sum_{n=3}^{\infty} \frac{\ln n}{n}$ diverges (to ∞).

So $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n} \overset{\text{note}}{=} \frac{\ln 2}{2} + \sum_{n=3}^{\infty} \frac{\ln n}{n}$ diverges (to ∞).

Now go up and mark the diverges box.

Ex 3 $\displaystyle\sum_{n=1}^{\infty} \frac{6+5n}{n^3}$ ☐ converges ☐ diverges.

Well,

$$\sum_{n=1}^{\infty} \frac{6+5n}{n^3} = \sum_{n=1}^{\infty} \left[\frac{6}{n^3} + \frac{5n}{n^3}\right] = ?$$

$$\Leftrightarrow \quad 6 \boxed{\sum_{n=1}^{\infty} \frac{1}{n^3}} + 5 \boxed{\sum_{n=1}^{\infty} \frac{1}{n^2}}.$$

p-series
$p = 3 > 1$
so converges

p-series
$p = 2 > 1$
so converges.

So $\displaystyle\sum_{n=1}^{\infty} \frac{6+5n}{n^3}$ converges (so go up and mark the converges box).

Closing Remarks

① When do we use the integral test?
When the 4 conditions are satisfied and $\int f(x)\,dx$ is "easy" to integrate.

② The main use of the Integral Test is to figure out what the

p-series, i.e. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$,

does. Section 11.4 uses the p-series (big time) in the Comparison Test and Limit Comparison Test.