§ 11.9 Representing a function as a power series

Ex 0 Can we represent the function

$$f(x) = \frac{1}{1-x}$$

as a power series?

Yes! How? With help of the Geometric Series:

$$\sum_{n=0}^{\infty} r^n \underset{=}{\text{work}} \frac{1}{1-r} \quad \text{when } |r| < 1.$$

So

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{when } |x| < 1.$$

Next Go through "Operations on Power Series" handout.

Now For Ex. 1-4 (to come):

- find a power series expansion for the given function $y = f(x)$

and

- say for which values of x it is valid (i.e., true/hold/etc).

We will do this by using known power series expansions and "Operations on Power Series."

Ex 1 $f(x) = \dfrac{x}{1+3x^2}$

Game plan: transform $\dfrac{x}{1+3x^2}$ into something so can use Geometric Series.

$$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n \quad \text{when } |r|<1. \quad \text{(GS)}.$$

Well

$$\frac{x}{1+3x^2} \overset{(A)}{=} x\left[\frac{1}{1+3x^2}\right] \overset{(A)}{=} x\left[\frac{1}{1-(-3x^2)}\right] \quad \text{GS w/ } r=-3x^2$$

$$\overset{GS}{=} x\left[\sum_{n=0}^{\infty}(-3x^2)^n\right]$$

$$\boxed{\text{Note} \quad (-3x^2)^n = (-3)^n (x^2)^n = (-1\cdot 3)^n x^{2n} = (-1)^n 3^n x^{2n}}$$

operations on power series

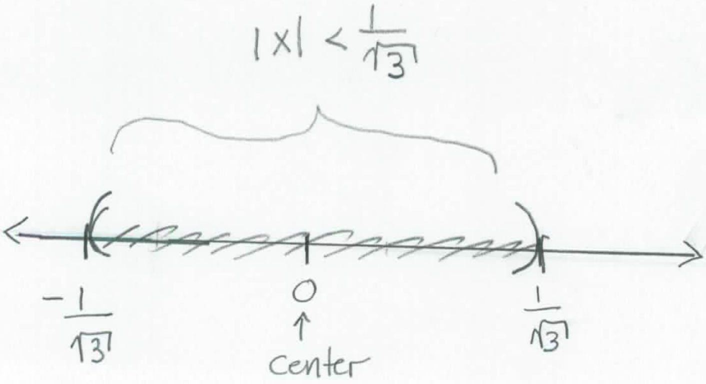
$$\overset{(A)}{=} x\left[\sum_{n=0}^{\infty}(-1)^n 3^n x^{2n}\right] = \sum_{n=0}^{\infty}(-1)^n 3^n x^{2n}\cdot x^1 = \sum_{n=0}^{\infty}(-1)^n 3^n x^{2n+1}.$$

The expansion is valid when .... we used geometric series (GS) ....

$$|r| \equiv |3x^2|<1 \iff 3|x^2|<1 \iff |x|^2<\frac{1}{3} \iff |x|<\frac{1}{\sqrt{3}}.$$

So//

$$\boxed{\frac{x}{1+3x^2} = \sum_{n=0}^{\infty}(-1)^n 3^n x^{2n+1} \quad \text{when } |x|<\frac{1}{\sqrt{3}}}$$



Center $= 0$

Radius $= \dfrac{1}{\sqrt{3}}$

Ex 2 $f(x) = \dfrac{1}{(1-x)^2}$.

Well

$$\frac{1}{(1-x)^2} = (1-x)^{-2} = D_x (1-x)^{-1} = D_x \left[\frac{1}{1-x} \right]$$ ← Geometric series "$r = x$"

$$\overset{GS}{=} D_x \left[\sum_{n=0}^{\infty} x^n \right]$$

Way 1: long

$$= D_x \left[1 + x + x^2 + x^3 + x^4 + \ldots \right]$$

$$= 0 + 1 + 2x + 3x^2 + 4x^3 + \ldots$$

$$= \sum_{n=1}^{\infty} n x^{n-1}$$

Beware: "lose" $n=0$ term b/c it's a constant.

Way 2: short

$$= \sum_{n=\boxed{0}}^{\infty} D_x x^n$$

why?

$$= \sum_{n=\boxed{1}}^{\infty} D_x x^n = \sum_{n=1}^{\infty} n x^{n-1}$$

both ways get

$$= \sum_{n=1}^{\infty} n x^{n-1} = 1x^0 + 2x^1 + 3x^2 + \ldots$$

$$\overset{so}{=} \sum_{n=0}^{\infty} (n+1) x^n.$$

Valid when the GS expansion used above in valid.

We used GS w/ $r = x$ so valid when $|x| < 1$. So

$$\boxed{\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1) x^n \quad \text{when } |x| < 1.}$$

Ex 3 $\quad f(x) = \tan^{-1} x$.

Hint: Is there a relation between $\tan^{-1} x$ and geometric series?

Yes, $\tan^{-1} x = \int \frac{1}{1+x^2}\,dx \quad (+C)$. Let's go ....

$$\tan^{-1} x = \int \frac{1}{1+x^2}\,dx = \int \frac{1}{1-(-x^2)}\,dx$$

Use Geometric Series with $r = -x^2$

$$\stackrel{GS}{=} \int \left[\sum_{n=0}^{\infty} (-x^2)^n \right] dx$$

valid when $|r| = |-x^2| < 1 \Leftrightarrow |x|^2 < 1 \Leftrightarrow |x| < 1$

operations on power series

$$= \sum_{n=0}^{\infty} \int (-1)^n x^{2n}\,dx$$

used $(-x^2)^n = (-1 \cdot x^2)^n = (-1)^n x^{2n}$

$$= \left[\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \right] + C$$

$$\tan^{-1} x = C + \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \pm \cdots \right]$$

to find C, let ~~$x = 17$~~ $x = 0$.

$$\underbrace{\tan^{-1} 0}_{\text{know } 0} = C + [0 - 0 + 0 - 0 \pm \cdots] \Rightarrow C = 0.$$

So

$$\boxed{\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{when } |x| < 1}$$

Remark: The $\boxed{\text{above}}$ expansion is also valid when $x = \pm 1$ but we need more theory to show that the "endpoints behave"

Ex 4 $f(x) = \int_0^x \arctan t \, dt$

Well, can use Ex 3.

Note, t, not x, why?

$$\int_0^x \arctan t \, dt \overset{(*)}{=} \int_0^x \left[\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{2n+1} \right] dt$$

By Ex 3. (*) Valid when $|t|<1$

operations on power series

$$= \sum_{n=0}^{\infty} \left[\int_0^x \frac{(-1)^n \, t^{2n+1}}{2n+1} \, dt \right]$$

$$= \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{2n+1} \int_0^x t^{2n+1} \, dt \right]$$

$$= \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{2n+1} \cdot \frac{t^{2n+2}}{2n+2} \Bigg|_{t=0}^{t=x} \right]$$

$$= \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{2n+1} \left(\frac{x^{2n+2}}{2n+2} - \frac{0^{2n+2}}{2n+2} \right) \right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2)} x^{2n+2}$$

$$= \frac{1}{1\cdot 2} x^2 - \frac{1}{3\cdot 4} x^4 + \frac{1}{5\cdot 6} x^6 - \frac{1}{7\cdot 8} x^8 +/- \cdots$$

$$= \sum_{n=1}^{\infty} \frac{?}{?} x^{2n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)(2n)} x^{2n}.$$

Valid?

$\int_0^x \cdots dt \longrightarrow$ $-|x|$ 0 $|x|$ $\rightarrow$ t is in here.

(*) is valid when $|t|<1$

so need $|x|<1$.

So

$$\boxed{\int_0^x \tan^{-1} t \, dt = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)(2n)} x^{2n} \quad \text{when } |x|<1.}$$