§ 11.8 - - Power Series

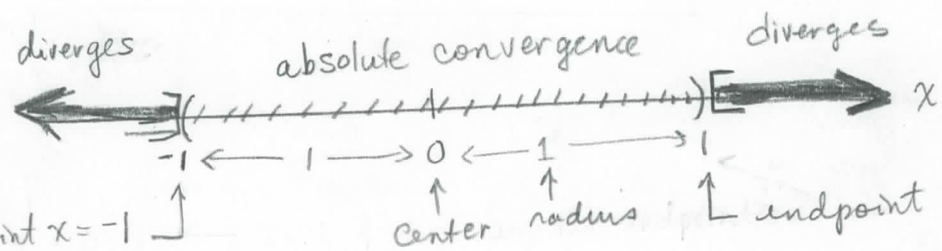
$\boxed{\text{b/c } \sum |t^n| = \sum |t|^n \text{ converges when } \underbrace{|\,|t|\,|}_{=|t|} < 1}$

Motivation

Recall Geometric Series $\sum_{n=1}^{\infty} t^n = \begin{cases} \text{converge absolutely} & \text{if } |t| < 1 \\ \text{diverges} & \text{if } |t| \geq 1 \end{cases}$

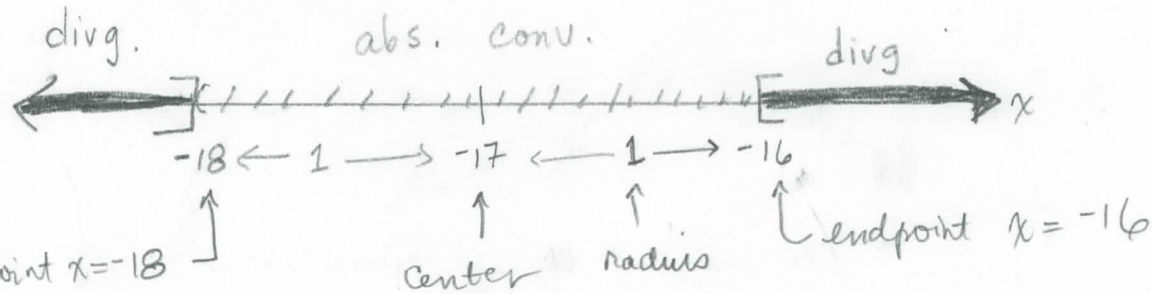
Ex 0a ($t = x$) $\sum_{n=1}^{\infty} x^n = \begin{cases} \text{absolutely convergent} & \text{if } |x| < 1 \\ \text{divergent} & \text{if } |x| \geq 1 \end{cases}$

$|x| < 1 \iff \underset{\text{center} = 0}{|x - 0|} < \underset{\text{radius} = 1}{1} \iff$ (the distance btw. x & 0) < 1.



Ex 0b ($t = x+17$) $\sum_{n=1}^{\infty} (x+17)^n = \begin{cases} \text{abs. conv.} & \text{if } |x+17| < 1 \\ \text{divg.} & \text{if } |x+17| \geq 1 \end{cases}$

$|x+17| < 1 \iff |x - \underset{\text{center} = -17}{(-17)}| < \underset{\text{radius} = 1}{1} \iff$ (the distance btw x & -17) < 1



Ex 0c

$t = 3x-6$

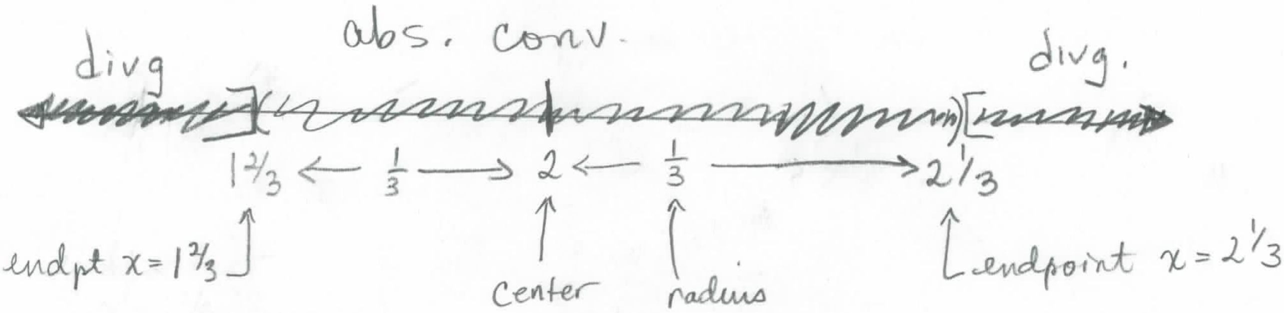
$$\sum_{n=17}^{\infty} (3x-6)^n = \begin{cases} \text{abs. conv.} & \text{if } |3x-6| < 1 \\ \text{divg.} & \text{if } |3x-6| \geq 1 \end{cases}$$

$n=17$ ← does this make a difference?

$$|3x-6|<1 \iff |3(x-2)|<1 \iff 3\,|x-2|<1 \iff |x-2|<\tfrac{1}{3}$$

$\iff$ (the distance btw x & 2) $< \frac{1}{3}$

center $= 2$ radius $= \frac{1}{3}$



Note

$$\sum_{n=17}^{\infty} (3x-6)^n = \sum_{n=17}^{\infty} [3(x-2)]^n$$

$$= \sum_{n=17}^{\infty} 3^n (x-2)^n$$

$$= \sum_{n=17}^{\infty} c_n (x-x_0)^n$$

$x_0 = 2 =$ center

$c_n = 3^n$.

Def. Let x_0 be a constant. Let c_n's be constants.

A power series in $(x-x_0)$, or

a power series centered about x_0,

is a series of the form

$$\boxed{\sum_{n=0}^{\infty} c_n (x-x_0)^n \quad . \qquad (*)}$$

Remark Note that

$$\sum_{n=0}^{\infty} c_n (x-x_0)^n = c_0 + c_1(x-x_0)^1 + c_2(x-x_0)^2 + c_3(x-x_0)^3 + \cdots .$$

Note that if you that $x = x_0$ in $(*)$, then

$$\sum_{n=0}^{\infty} c_n (x-x_0)^n = \boxed{}$$

and so

$$\sum_{n=0}^{\infty} c_n (x-x_0)^n \quad \text{converges if } x = x_0 .$$

The Question

For what values of x does $\sum_{n=0}^{\infty} c_n (x-x_0)^n$

(1) converge absolutely

(2) converge conditionally

(3) diverge

?

The Answer

....

Theorem. Consider a power series

$$\boxed{\sum_{n=0}^{\infty} c_n (x-x_0)^n \quad . \qquad (*)}$$

This power series in $(*)$ has a

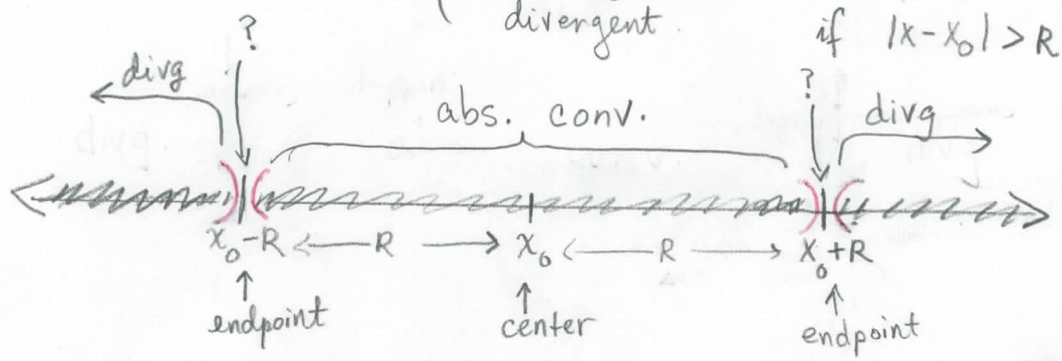
Radius of Convergence R, where $0 \le R \le \infty$,

such that precisely 1 of the following 3 cases happens.

(1) $0 < R < \infty$, in which case

$$\sum_{n=0}^{\infty} c_n (x-x_0)^n = \begin{cases} \text{abs. conv.} & \text{if } |x-x_0| < R \\ \text{anything can happen} & \text{if } |x-x_0| = R \\ \text{divergent} & \text{if } |x-x_0| > R \end{cases}$$

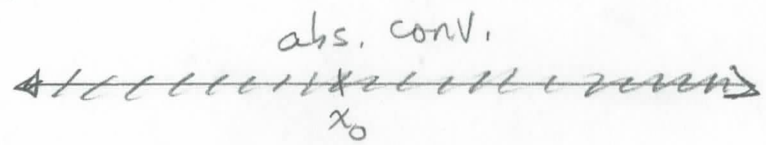
endpoints $\boxed{\begin{array}{l} x = x_0 + R \\ x = x_0 - R \end{array}}$



at the 2 endpoints $x = x_0 \pm R$, the power series $(*)$ can be abs. conv., or cond. conv., or divg.

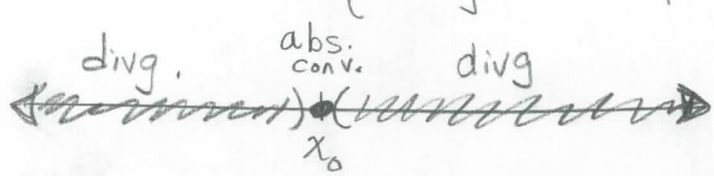
(2) $R = \infty$, in which case

$\sum_{n=0}^{\infty} c_n (x-x_0)^n$ is abs. conv. for all x.



(3) $R = 0$, in which case

$$\sum_{n=0}^{\infty} c_n (x-x_0)^n = \begin{cases} \text{abs. conv.} & \text{if } x = x_0 \\ \text{divg.} & \text{if } x \ne x_0 \end{cases}$$



Continued ... one more definition

The power series in (*) also has its

interval of convergence I.

This interval of convergence consists of all x's so that

$$\sum_{n=0}^{\infty} c_n (x-x_0)^n \qquad (*)$$

converges, either absolutely or conditionally.

So for each of the 3 cases we have the following.

(1) $0 < R < \infty$.

I is $(x_0-R,\ x_0+R)$ or $[x_0-R,\ x_0+R]$ or $(x_0-R,\ x_0+R]$ or $[x_0-R,\ x_0+R)$.

How do we know which 1 of the 4? We have to "check the endpoints".

(2) $R = \infty$. $I = (-\infty, \infty) \overset{\text{i.e.}}{=} \mathbb{R}$

(3) $R = 0$. $I =$ the point $x_0 \overset{\text{i.e.}}{=} \{x_0\} \overset{\text{i.e.}}{=} [x_0, x_0]$

Game Plan

① Given the power series

$$\boxed{\sum_{n=0}^{\infty} c_n (x-x_0)^n \qquad (\star)}$$

② We want to say when the power series in ($\star$) is: abs. conv., cond. conv., divg.

$|x-x_0|<R$ → abs. conv.; $|x-x_0|>R$ → divg.

$$\boxed{\begin{gathered}|x-x_0|=R\\ \Updownarrow \\ x=x_0\pm R\end{gathered}}$$

For this we make a "number line chart"

$$\longleftarrow \underset{x_0}{|} \longrightarrow x$$

③ So we need to find R. How do we do this?

④ We apply the Ratio or Root Test to

$$\sum_{n=0}^{\infty} \underbrace{|c_n (x-x_0)^n|}_{a_n} \, .$$ ← note absolute value sign

So.

Ratio Test $$\rho = \lim_{n\to\infty} \frac{|c_{n+1}(x-x_0)^{n+1}|}{|c_n (x-x_0)^n|} = \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| \left|\frac{(x-x_0)^{n+1}}{(x-x_0)^n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| |x-x_0|^1 = |x-x_0| \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|$$

Root Test $$\rho = \lim_{n\to\infty} \left[|c_n(x-x_0)^n|\right]^{1/n} = \lim_{n\to\infty} |c_n|^{1/n} \left[|x-x_0|^n\right]^{1/n}$$

$$= \lim_{n\to\infty} |c_n|^{1/n} |x-x_0| = |x-x_0| \lim_{n\to\infty} |c_n|^{1/n} .$$

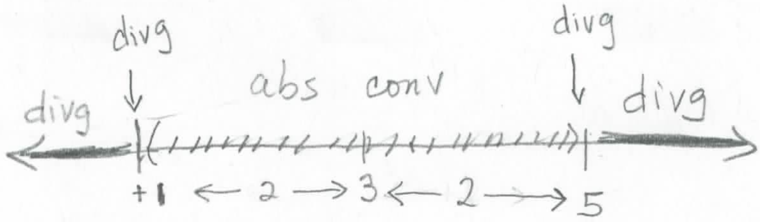
⑤ Then

$$\sum_{n=0}^{\infty} c_n (x-x_0)^n = \begin{cases} \text{abs. conv.} & \text{if } \rho<1 \\ \text{any thing can happen} & \text{if } \rho = 1 \\ \text{divg.} & \text{if } \rho>1 \end{cases}$$

$\boxed{\text{use this to find R}}$

⑥ Do you see there is nothing new to memorize if you understand previous ideas?

$\boxed{\text{Ex. 1}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{2^n}$



Root test to $\displaystyle\sum \left| \frac{(x-3)^n}{2^n} \right|$.

no "n" so this is a constant w/ respect to n

$$\rho = \lim_{n\to\infty} \left| \frac{(x-3)^n}{2^n} \right|^{1/n} = \lim_{n\to\infty} \frac{|x-3|}{2} = \frac{|x-3|}{2}$$

- abs conv when $\rho = \dfrac{|x-3|}{2} < 1 \iff |x-3| < 2$
- divg. when $\rho = \dfrac{|x-3|}{2} > 1 \iff |x-3| > 2$

$\Big\}$ radius of conv is 2

center is 3

Check endpoints $x = 1, 5$

$x=1 \Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{2^n} = \sum_{n=1}^{\infty} \frac{(-2)^n}{2^n} = \sum_{n=1}^{\infty} (-1)^n$ divg (osc)

$x=5 \Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{2^n} = \sum_{n=1}^{\infty} \frac{2^n}{2^n} = \sum_{n=1}^{\infty} 1$ divg (∞)

$\Rightarrow$ So radius of conv. $= 2$ and interval of conv $= (1, 5)$.

$\boxed{\text{Ex 2}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(x+17)^n}{n!}$ i.e. $\displaystyle\sum_{n=1}^{\infty} \frac{(x- {}^-17)^n}{n!}$ ← center $x_0 = {}^-17$

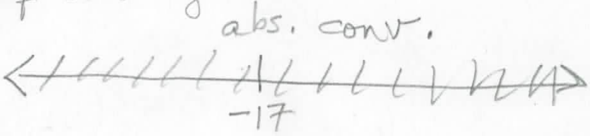
Ratio Test (why? b/c factorial) to $\displaystyle\sum \left| \frac{(x+17)^n}{n!} \right|$.

$$\rho = \lim_{n\to\infty} \left| \frac{(x+17)^{n+1}}{(n+1)!} \cdot \frac{n!}{(x+17)^n} \right| \overset{\text{algebra}}{\underset{(A)}{=}} \lim_{n\to\infty} \frac{n!}{(n+1)!} \, \frac{|x+17|^{n+1}}{|x+17|^n}$$

$$\overset{(A)}{=} \lim_{n\to\infty} \frac{|x+17|^1}{n+1} \overset{\text{trick}}{=} |x+17| \lim_{n\to\infty} \frac{1}{n+1} \overset{\text{trick}}{=} |x+17| \cdot 0 .$$

- absolute convergence when $\rho = |x+17| \cdot 0 < 1 \iff x \in (-\infty, \infty)$ i.e. $x \in \mathbb{R}$.

So radius of convergence $= \infty$ and interval of convergence $= (-\infty, \infty)$



Ex 3 $\sum_{n=0}^{\infty} n!\,(x+2)^n \quad \underline{\text{note}} \quad \sum_{n=0}^{\infty} n!\,(x-(-2))^n$

center is -2.

Ratio Test to $\sum |n!\,(x+2)^n|$.

$$\rho = \lim_{n\to\infty} \left| \frac{(n+1)!\,(x+2)^{n+1}}{n!\,(x+2)^n} \right| = \lim_{n\to\infty} \frac{|x+2|^{n+1}}{|x+2|^n} \cdot \frac{(n+1)!}{n!}$$

$$= \lim_{n\to\infty} |x+2| \cdot (n+1) \underset{\text{See Ex 2}}{\overset{\text{trick}}{=}} |x+2| \lim_{n\to\infty}(n+1) = ?$$

Well

If $|x+2| \neq 0$, i.e. if $x \neq -2$, then

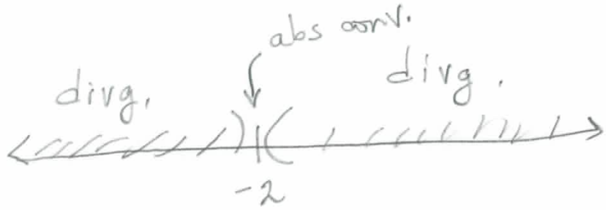
$$\rho = |x+2| \lim_{n\to\infty}(n+1) = \underbrace{|x+2|}_{\neq 0} \cdot \infty = \infty > 1$$

diverges.

If $|x+2| = 0$, ie if $x = -2$, then

$$\sum_{n=0}^{\infty} n!\,(x+2)^n = 0! + (1!\cdot 0) + (2!\cdot 0) + (3!\cdot 0) = 0! = 1$$

So have



So radius of convergence is 0 and "interval" of convergence $= \{2\} = [2,2]$

$\boxed{\text{Ex 4}}$ $\sum_{n=1}^{\infty} \frac{(3x-6)^n}{n}$. What is the center (ie. x_0)? Let's do some algebra.

$$\sum_{n=1}^{\infty} \frac{(3x-6)^n}{n} = \sum_{n=1}^{\infty} \frac{[3(x-2)]^n}{n} = \sum_{n=1}^{\infty} \frac{3^n}{n}(x-2)^n$$

So the center is $x_0 = +2$.

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Ratio Test to $\sum \left| \overbrace{\frac{(3x-6)^n}{n}}^{a_n} \right| = \sum \left| \frac{3^n (x-2)^n}{n} \right|$

$$\rho = \lim_{n\to\infty} \left| \frac{3^{n+1} \quad (x-2)^{n+1}}{n+1} \cdot \frac{n}{3^n (x-2)^n} \right|$$

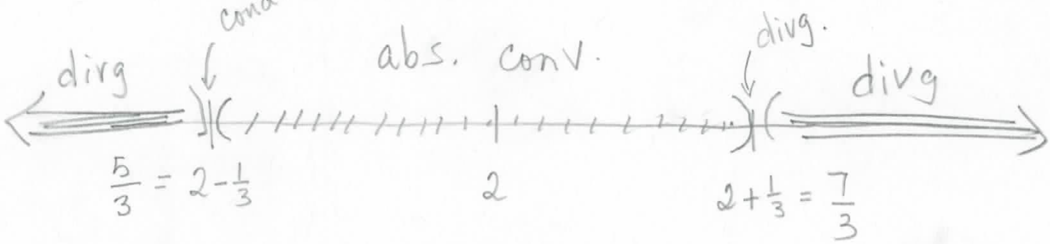
trick

$$= \lim_{n\to\infty} 3\,|x-2| \quad \frac{n}{n+1} = 3\,|x-2|$$

abs. conv when $\rho = 3|x-2| < 1 \iff |x-2| < \frac{1}{3}$

divg when $\rho = 3|x-2| > 1 \iff |x-2| > \frac{1}{3}$

radius of conv is $1/3$



Check endpts

$x = \frac{7}{3} \Rightarrow \sum_{n=1}^{\infty} \frac{(3x-6)^n}{n} \overset{x=\frac{7}{3}}{=\!=} \sum_{n=1}^{\infty} \frac{(1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ ← divg harmonic series

$x = \frac{5}{3} \Rightarrow \sum_{n=1}^{\infty} \frac{(3x-6)^n}{n} \overset{x=\frac{5}{3}}{=\!=} \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ ← cond. conv. A.S.T. ⊕ harmonic series

$\Rightarrow$ So radius of conv. $= \frac{1}{3}$

& interval of conv $= \left[\frac{5}{3}, \frac{7}{3}\right)$

includes $5/3$ omit $7/3$