## § 11.6 Ratio and Root Test

We will differ some from the book.

Let's look at the Series Summary Sheet handout.

$$\boxed{\begin{array}{l}\text{Arbitrary-Termed Series Test} \\ \sum a_n \quad \text{where} \quad -\infty < a_n < \infty \quad \forall\, n \in \mathbb{N}.\end{array}}$$

Ratio Test   Let $\rho := \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$

Root Test   Let $\rho := \lim_{n \to \infty} \sqrt[n]{|a_n|} \overset{\text{note}}{=} \lim_{n \to \infty} |a_n|^{\frac{1}{n}}$.

Then

$0 \le \rho < 1 \Rightarrow \sum a_n$ converges absolutely

$\rho = 1 \Rightarrow$ test is inconclusive

$1 < \rho \le \infty \Rightarrow \sum a_n$ diverges.

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Question 1  Can you conclude conditional convergence from the Ratio/Root Test?

NO - why so ??

Remark 2  Often use the Ratio Test if have factorials (eg $(n!)$ or $(2n)!$).

Why ... see next example.

Example 3  simpify $\frac{(2n)!}{(2(n+1))!} = ?$

$$\frac{(2n)!}{(2(n+1))!} = \frac{(2n)!}{(2n+2)!} = \frac{(2n)!}{(2n)!\,(2n+1)(2n+2)} = \frac{1}{(2n+1)(2n+2)}$$

Ex 4 $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{(2n)!}$

- ☐ absolutely convergent
- ☐ conditionally convergent
- ☐ divergent.

So here we have $\sum a_n$ with $a_n = \dfrac{(-1)^n 3^n}{(2n)!}$ and $|a_n| = \dfrac{3^n}{(2n)!}$

$\boxed{*}$ Is it abs. conv.? Consider $\sum |a_n| = \sum \dfrac{3^n}{(2n)!}$.

Factorial in $|a_n| = \dfrac{3^n}{(2n)!}$ so try ratio test.

$$\rho := \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{3^{n+1}}{[2(n+1)]!} \cdot \frac{(2n)!}{3^n}$$ ← collect up like terms

$$= \lim_{n\to\infty} \frac{3^{n+1}}{3^n} \cdot \frac{(2n)!}{[2(n+1)]!} \underset{\text{see Ex 3}}{=} \lim_{n\to\infty} (3) \left[\frac{1}{(2n+1)(2n+2)}\right] = 0 < 1$$

So by the ratio test, $\sum \dfrac{(-1)^n 3^n}{(2n)!}$ is abs conv.

Ex 5 $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n^2}$

- ☐ absolutely convergent
- ☐ conditionally convergent
- ☐ divergent

So here we have $\sum a_n$ with $a_n = \dfrac{(-1)^n 3^n}{n^2}$ so $|a_n| = \dfrac{3^n}{n^2}$.

Let's use the ratio test since we foresee cancellation heaven when we do $\left|\dfrac{a_{n+1}}{a_n}\right|$.

collect up like terms

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)^2} \cdot \frac{n^2}{3^n} = \lim_{n\to\infty} \frac{3^{n+1}}{3^n} \frac{n^2}{(n+1)^2}$$

$$= \lim_{n\to\infty} 3 \cdot \left(\frac{n}{n+1}\right)^2 = 3 \equiv \rho > 1.$$

So, by the Ratio Test, $\sum \dfrac{(-1)^n 3^n}{n^2}$ is divergent.

Remark 6  Useful for Root test is

$$\boxed{\lim_{n\to\infty} n^{1/n} = 1.} \qquad (*)$$

To show (*) is true, here are some hints.

- $\lim_{n\to\infty} n^{1/n} = $ " $\cancel{\infty^0}$ ", an indeterminate form, handled by:
- Take $\ln(n^{1/n})$, use L'Hôpital's rule, then expontiate back.
- $\lim_{n\to\infty} \ln(n^{1/n}) = \lim_{n\to\infty} \frac{\ln n}{n} \underset{L'H}{\overset{\frac{\infty}{\infty}}{=}} \lim_{n\to\infty} \frac{1/n}{1} = 0.$
- $n^{1/n} = e^{\ln(n^{1/n})} \xrightarrow{n\to\infty} e^0 = 1.$  Not so bad...

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Ex 7. $\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n^2}$  ☐ abs. conv.
☐ cond. conv.
☐ divergent.

Hey, this is Ex. 5, which we did w/ Ratio Test. Well, let's have some fun and see how it goes w/ the Root Test.

Root Test

$$\rho = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left[\frac{3^n}{n^2}\right]^{1/n} = \lim_{n\to\infty} \frac{3^{n\cdot\frac{1}{n}}}{n^{2\cdot\frac{1}{n}}}$$

$$= \lim_{n\to\infty} \frac{3}{(n^{1/n})^2} \underset{\text{Remark 6}}{=} \frac{3}{1^2} = 3 > 1$$

So by the Root test, $\sum \frac{(-1)^n 3^n}{n^2}$ divergences.

**Remark 8** Often use Root Test if have

$$(\#)^n$$

for then

$$\rho := \lim_{n \to \infty} \left[ |\#|^n \right]^{1/n} = \lim_{n \to \infty} |\#| .$$

**Ex 9** $\displaystyle\sum_{n=1}^{\infty} \frac{2^{3n+1}}{n^n}$

- ☐ absolutely convergent
- ☐ conditionally convergent
- ☐ divergent

$\boxed{9.1}$ Why can you cross out conditionally convergent?

$\boxed{9.2}$ Let's get started. Well :

$$\frac{2^{3n+1}}{n^n} = \frac{2^{3n} \cdot 2^1}{n^n} = \frac{2 \cdot (2^3)^n}{n^n} = \frac{2(8)^n}{n^n} = 2 \cdot \left(\frac{8}{n}\right)^n .$$

So let's use root test! Why ... let's see...

Well: $\displaystyle (|a_n|)^{1/n} = \left( \frac{2^{3n+1}}{n^n} \right)^{1/n} = \frac{2^3 \cdot 2^{1/n}}{n} \xrightarrow{n \to \infty} \cancel{\frac{8 \cdot 1}{\text{“}\infty\text{”}}} = 0 <$

So by ratio test, $\displaystyle\sum \frac{2^{3n+1}}{n^n}$ is abs. convergent.

**Warning 10** If doing Ratio or Root Test & you get $\rho = 1$, then the test is inconclusive and you need to use a different test.