# Alternating Series Estimation Theorem. (ASET)

Consider an alternating series $\sum_{n=1}^{\infty} (-1)^n u_n$ satisfying

(i) $u_n > u_{n+1} > 0$ (ii) $\lim_{n\to\infty} u_n = 0$.

The AST tells us that $\sum_{n=1}^{\infty} (-1)^n u_n \equiv L$ converges.

So

$$\underbrace{\sum_{n=1}^{\infty} (-1)^n u_n}_{\text{want to find}} = \underbrace{\sum_{n=1}^{N} (-1)^n u_n}_{\text{can find}} + \underbrace{\sum_{n=N+1}^{\infty} (-1)^n u_n}_{\text{hard to find}}$$

So

$$\sum_{n=1}^{\infty} (-1)^n u_n \overset{N \text{ big}}{\approx} \sum_{n=1}^{N} (-1)^n u_n$$

ASET tells us:

$$\left| \sum_{n=1}^{\infty} (-1)^n u_n - \sum_{n=1}^{N} (-1)^n u_n \right| = \left| \sum_{n=N+1}^{\infty} (-1)^n u_n \right| \overset{\text{ASET}}{\leq} u_{N+1}$$

cool...

$$\sum_{n=1}^{\infty} (-1)^n u_n \approx \sum_{n=1}^{N} (-1)^n u_n$$

within an error of $u_{N+1}$

→

ASET Example: Consider the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{4}{2n+1} \left[ \left(\tfrac{1}{2}\right)^{2n+1} + \left(\tfrac{1}{3}\right)^{2n+1} \right]. \quad (*)$$

(1.1) Is (*) an alternating series? That is, can (*) be expressed as $\sum (-1)^n u_n$ or $\sum (-1)^{n+1} u_n$ for some $u_n > 0$?

Yes, (*) is of the form $\boxed{\sum (-1)^n u_n}$ where

$$\boxed{u_n = \frac{4}{2n+1}\left[\frac{1}{2^{2n+1}} + \frac{1}{3^{2n+1}}\right]} \underset{\text{algebra}}{=} \frac{4}{2n+1}\left[\frac{1}{2}\cdot\frac{1}{4^n} + \frac{1}{3}\cdot\frac{1}{9^n}\right] > 0.$$

(1.2) Does (*) converge? Let's check conditions of the AST.

- $u_n$ decreasing? i.e. $u_n > u_{n+1}$? ✓ clear
- $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{4}{2n+1}\left[\frac{1}{2}\cdot\left(\frac{1}{4}\right)^n + \frac{1}{3}\left(\frac{1}{9}\right)^n\right] = 0\,[0+0] = 0$ ✓

Recall, $\lim_{n\to\infty} r^n = 0$ if $|r| < 1$

So by the AST, (*)

(1.3) What is the smallest $N$ so that the ASET guarantees that

$$\left| \left[\sum_{n=0}^{\infty} (-1)^n u_n\right] - \left[\sum_{n=0}^{N} (-1)^n u_n\right] \right| < 0.0005.$$

Well, ASET tells us that

$$\left| \left[\sum_{n=0}^{\infty} (-1)^n u_n\right] - \left[\sum_{n=0}^{N} (-1)^n u_n\right] \right| \le u_{N+1}.$$

Note that

$$u_{N+1} = \frac{4}{2(N+1)+1}\left[\frac{1}{2^{2(N+1)+1}} + \frac{1}{3^{2(N+1)+1}}\right] = \frac{4}{2N+3}\left[\frac{1}{2^{2N+3}} + \frac{1}{3^{2N+3}}\right].$$

Let's make a chart.

2 →

11.36

| $N$ | $N+1$ | $2N+3$ | $u_{N+1} = \frac{4}{2N+3}\left[\frac{1}{2^{2N+3}} + \frac{1}{3^{2N+3}}\right]$ |
|---|---|---|---|
| 0 | 1 | 3 | $u_1 = \frac{4}{3}\left[\frac{1}{2^3} + \frac{1}{3^3}\right] \approx 0.216049 \not< 0.0005$ |
| 1 | 2 | 5 | $u_2 = \frac{4}{5}\left[\frac{1}{2^5} + \frac{1}{3^5}\right] \approx 0.02829 \not< 0.0005$ |
| 2 | 3 | 7 | $u_3 = \frac{4}{7}\left[\frac{1}{2^7} + \frac{1}{3^7}\right] \approx 0.00473 \not< 0.0005$ |
| 3 | 4 | 9 | $u_4 = \frac{4}{9}\left[\frac{1}{2^9} + \frac{1}{3^9}\right] \approx 8.906\,E^{-4} = \frac{8.906}{1\underbrace{0000}_{4}} = \frac{0.0008.906}{10000.} = 0.0008906 \not< 0.0005$ |
| 4 | 5 | 11 | $u_5 = \frac{4}{11}\left[\frac{1}{2^{11}} + \frac{1}{3^{11}}\right] \cong 1.796\,E^{-4} = \frac{1.796}{10000} = \frac{0.0001.796}{10000.} = 0.0001796 < 0.0005$ |

Thus

$$\left|\sum_{n=1}^{\infty} (-1)^n u_n - \sum_{n=1}^{4} (-1)^n u_n\right| \underset{\text{ASET}}{\le} u_5 < 0.0005$$

but

$$\left|\sum_{n=1}^{\infty} (-1)^n u_n - \sum_{n=1}^{3} (-1)^n u_n\right| \le u_4 \not< 0.0005.$$

So $\boxed{N = 4}$.