Pre § 11.10 and 11.11 : DO , Taylor Series and Polynomials

- Work through the worksheet "Taylor/Maclaurin Polynomials Warm Up" which is posted on course Ch 11 homework page.
- Do homework problems 2, 4, 5, 6 from this worksheet.

§ 11.10 and 11.11 Taylor/Maclaurin Series/Polynomials.

First Go over page 2 of course handout "Taylor/Maclaurin Polynomials and Series"

Now For the below (to come) Taylor Example 1 and 2, for the given function $y = f(x)$ and given center x_0, find the following.

① The N^{th}-order Taylor polynomial of $y = f(x)$ at x_0 for $N = 0, 1, 2, 3, 4$. So find $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$ and $P_4(x)$.

② Find a general formula for the n^{th} Taylor coefficient of $y = f(x)$ at x_0. So find c_n.

③ The Taylor series of $y = f(x)$ at x_0, in closed form using the sigma (Σ) notation. So find $P_\infty(x)$.

Taylor Example 1.

$$f(x) = \frac{1}{1-x} \quad \text{and} \quad x_0 = 0.$$

① $$P_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n$$

$$= f(0) + f^{(1)}(0)\,x^1 + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(N)}(0)}{N!}x^N.$$

=

note pattern.

$$f^{(0)}(x) \overset{\text{i.e.}}{=} f(x) = (1-x)^{-1} \qquad f^{(0)}(0) = 1 = 0!$$

$$f^{(1)}(x) = (1-x)^{-2} \qquad f^{(1)}(0) = 1 = 1!$$

$$f^{(2)}(x) = 2(1-x)^{-3} \quad \Rightarrow \quad f^{(2)}(0) = 2 = 2!$$

$$f^{(3)}(x) = 2\cdot 3(1-x)^{-4} \qquad f^{(3)}(0) = 3! = 3!$$

$$f^{(4)}(x) = 2\cdot 3\cdot 4(1-x)^{-5} \qquad f^{(4)}(0) = 4! = 4!$$

=

$$P_0(x) = 1$$

$$P_1(x) = 1 + 1x$$

$$P_2(x) = 1 + 1x + \frac{2}{2!}x^2$$

$$P_3(x) = 1 + 1x + \frac{2}{2!}x^2 + \frac{3!}{3!}x^3$$

$$P_4(x) = 1 + 1x + \frac{2}{2!}x^2 + \frac{3!}{3!}x^3 + \frac{4!}{4!}x^4$$

=

Now just simplify

So

$$\begin{cases} P_0(x) = 1 \\ P_1(x) = 1 + x \\ P_2(x) = 1 + x + x^2 \\ P_3(x) = 1 + x + x^2 + x^3 \\ P_4(x) = 1 + x + x^2 + x^3 + x^4 . \end{cases}$$

(2) $$C_n = \frac{f^{(n)}(x_0)}{n!} = \frac{f^{(n)}(0)}{n!}$$

$$\left.\begin{aligned} f(x) &= (1-x)^{-1} &&= 0!\,(1-x)^{-1} \\ f'(x) &= (1-x)^{-2} &&= 1!\,(1-x)^{-2} \\ f^{(2)}(x) &= 2(1-x)^{-3} &&= 2!\,(1-x)^{-3} \\ f^{(3)}(x) &= 2\cdot 3\,(1-x)^{-4} &&= 3!\,(1-x)^{-4} \\ f^{4}(x) &= \underbrace{2\cdot 3\cdot 4\,(1-x)^{-5}} &&= 4!\,(1-x)^{-5} \end{aligned}\right] \Rightarrow \underbrace{\begin{aligned} f^{(0)}(0) &= 1 = 0! \\ f^{(1)}(0) &= 1! \\ f^{(2)}(0) &= 2! \\ f^{(3)}(0) &= 3! \\ f^{(4)}(0) &= 4! \end{aligned}}$$

already did
do you see the pattern?

do you see the pattern?

So $f^{(n)}(0) = n!$ for $n = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, \ldots\ldots$

$$C_n = \frac{f^{(n)}(0)}{n!} = \frac{n!}{n!} = 1 \quad \text{for } n = 0, 1, 2, \ldots.$$

$$\boxed{C_n = 1 \quad \text{for } n = 0, 1, 2, \ldots.}$$

(3) $$P_\infty(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \underset{\text{here}}{\overset{x_0 = 0}{=}} \sum_{n=0}^{\infty} \boxed{\frac{f^{(n)}(0)}{n!}}\, x^n = \boxed{\sum_{n=0}^{\infty} 1 \cdot x^n}$$

$$\boxed{P_\infty(x) = \sum_{n=0}^{\infty} x^n}$$

note $1 = x^0$

compare

In open form, $P_\infty(x) = \underbrace{1}_{x^0} + x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots$

<u>Aside</u> Let's think about this example some more.

We started with

$$f(x) = \frac{1}{1-x} \quad \text{and} \quad x_0 = 0$$

and found it's Taylor series about $x_0 = 0$ to be

$$P_\infty(x) = \sum_{n=0}^{\infty} x^n .$$

From the geometric series we know

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \text{when } |x| < 1.$$

<u>So</u>

$$f(x) = P_\infty(x) \qquad \text{when } |x| < 1.$$

<u>Neat!</u> <u>Amazing!</u> More on this idea to come.

Taylor Example 2 $\quad f(x) = \sin x \quad$ and $\quad x_0 = \pi$

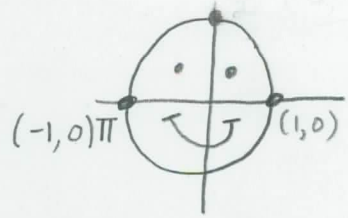
in calculus we always work in radians not degrees! Why?

① $P_N(x) = \displaystyle\sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = \sum_{n=0}^{N} \frac{f^{(n)}(\pi)}{n!}(x-\pi)^n$

starts to repeat

$$\left[\begin{array}{l} f^{(0)}(x) = \sin x \\ f^{(1)}(x) = \cos x \\ f^{(2)}(x) = -\sin x \\ f^{(3)}(x) = -\cos x \\ f^{(4)}(x) = \sin x \end{array}\right] \Rightarrow \left[\begin{array}{l} f^{(0)}(\pi) = 0 \\ f^{(1)}(\pi) = -1 \\ f^{(2)}(\pi) = 0 \\ f^{(3)}(\pi) = +1 \\ f^{(4)}(\pi) = 0 \end{array}\right]$$

starts to repeat



helpful

$P_4(x) = f^{(0)}(\pi) + f^{(1)}(\pi)(x-\pi) + \dfrac{f^{(2)}(\pi)}{2!}(x-\pi)^2 + \dfrac{f^{(3)}(\pi)}{3!}(x-\pi)^3 + \dfrac{f^{(4)}(\pi)}{4!}(x-\pi)^4$

$P_0(x) = 0 \qquad\qquad = 0$

$P_1(x) = 0 - (x-\pi) \qquad\qquad = -(x-\pi)$

$P_2(x) = 0 - (x-\pi) + 0 \qquad\qquad = -(x-\pi)$

$P_3(x) = 0 - (x-\pi) + 0 + \frac{1}{3!}(x-\pi)^3 \qquad\qquad = -(x-\pi) + \frac{1}{3!}(x-\pi)^3$

$P_4(x) = 0 - (x-\pi) + 0 + \frac{1}{3!}(x-\pi)^3 + 0 \qquad = -(x-\pi) + \frac{1}{3!}(x-\pi)^3$

Why they are called N^{th} <u>order</u> Taylor polynomials instead of N^{th} <u>degree</u> Taylor polynomials

2 → [The <u>degree</u> of the N^{th} <u>order</u> Taylor polynomial] $\leq N$.

② $c_n = \dfrac{f^{(n)}(x_0)}{n!} = \dfrac{f^{(n)}(\pi)}{n!}$

$f^{(n)}(\pi) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \pm 1 & \text{if } n \text{ is odd} \end{cases}$

so

$c_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{\pm 1}{n!} & \text{if } n \text{ is odd} \end{cases}$

yek... lets do ③ first

(3) From (1) we see

$$P_\infty(x) = -(x-\pi)^1 + \frac{1}{3!}(x-\pi)^3 - \frac{1}{5!}(x-\pi)^5 + \frac{1}{7!}(x-\pi)^7 - \frac{1}{9!}(x-\pi)^9 \pm \cdots$$

$\Rightarrow P_\infty(x)$ looks like $\displaystyle\sum_{\text{over odd } n} (\pm 1)\frac{1}{n!}(x-\pi)^n$ ← clean up!

Note

$\{2n\}_{n=0}^{\infty} = \{0, 2, 4, 6, \ldots\}$ ← even #'s

$n=0$, $n=1$, $n=2$, $n=3$

$\{2n+1\}_{n=0}^{\infty} = \{1, 3, 5, 7, \ldots\}$ ← odd #'s

$$\Rightarrow P_\infty(x) = \sum_{n=\boxed{?}}^{\infty} (\pm 1)\frac{1}{(2n+1)!}(x-\pi)^{2n+1}$$

want to "start" with $(2n+1) = 1$ so want to start with $n=0$

$$\Rightarrow P_\infty(x) = \sum_{n=0}^{\infty} (\pm 1)\frac{1}{(2n+1)!}(x-\pi)^{2n+1}$$

how should this look? +1 or −1? Let's make a chart to see

n	$2n+1$	$+1$ or -1		
0	1	−1	$=(-1)^{\text{odd \#}}$	$=(-1)^{n+1}$
1	3	1	$=(-1)^{\text{even\#}}$	$=(-1)^{n+1}$
2	5	−1	$=(-1)^{\text{odd \#}}$	$=(-1)^{n+1}$
3	7	1	$=(-1)^{\text{even\#}}$	$=(-1)^{n+1}$
⋮	⋮			⋮

$$\Rightarrow \boxed{P_\infty(x) = \sum_{n=0}^{\infty} (-1)^{n+1}\frac{1}{(2n+1)!}(x-\pi)^{2n+1}} \overset{\text{or}}{=} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}(x-\pi)^{2n+1}$$

(2) Find a general formula for $c_n \equiv \dfrac{f^{(n)}(x_0)}{n!}$.

We have already noted:

$$c_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{\pm 1}{n!} & \text{if } n \text{ is odd} \end{cases}.$$

In (3) we found

$$P_\infty(x) = \sum_{k=0}^{\infty} c_{k} (x-\pi)^{k}$$

these always match up.

$$P_\infty(x) = \sum_{n=0}^{\infty} \boxed{\frac{(-1)^{n+1}}{(2n+1)!}} (x-\pi)^{2n+1}$$

here must be c_{2n+1}

So:

even $\quad c_{2n} = 0 \qquad n = 0, 1, 2, \ldots \quad (\text{so } 2n = 0, 2, 4, \ldots)$

odd. $\quad c_{2n+1} = \dfrac{(-1)^{n+1}}{(2n+1)!} \qquad n = 0, 1, 2, \ldots \quad (\text{so } 2n+1 = 1, 3, 5, \ldots$

Theorem If f has a power series representation (expansion) at $x_0 = a$, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad \text{for } |x-a| < R$$

then its coefficients are

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

In short, if a function can be represented by a power series then that power series is the function's Taylor series!

Read pages 3 and 4 of Taylor / Maclaurin Poly. & Series handout.

Revisit Taylor Example 2

The Taylor series of

$$f(x) = \sin x$$

about $x_0 = \pi$ is

$$P_\infty(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} (x-\pi)^{2n+1} .$$

We already did.

Now show that

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} (x-\pi)^{2n+1}$$

for all $x \in (-\infty, \infty)$.

Sol'n Know

$$f(x) = P_N(x) + R_N(x) .$$

We want to show that

$$\lim_{N \to \infty} R_N(x) = 0$$

or equivalently

$$\lim_{N \to \infty} |R_N(x)| = 0$$

for all $-\infty < x < \infty$.

So fix an $x \in (-\infty, \infty)$. The BIG Theorem says that for some c between x and $\pi = x_0$:

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!} (x-\pi)^{N+1}$$

Look back at Taylor Example 2

$$f(x) = \sin x$$

$f^{(N+1)}(x)$ is either $\cos x$, $-\cos x$, $\sin x$, or $-\sin x$,

so $|f^{(N+1)}(c)| \le 1$.

$$|R_N(x)| = |f^{(N+1)}(c)| \frac{1}{(N+1)!} |x-\pi|^{N+1}$$

$$\downarrow$$

$$\leq 1 \frac{|x-\pi|^{N+1}}{(N+1)!}$$

Also see homework assignment #10.2 11.27!

$$\Rightarrow$$

$$|R_N(x)| \leq \frac{|x-\pi|^{N+1}}{(N+1)!} \xrightarrow[\text{why?}]{N\to\infty} 0$$

by n^{th} term test.

$$\Downarrow$$

$$\sum \frac{|x-\pi|^{N+1}}{(N+1)!}$$

☒ absolute convergent ← by ratio test
☐ ~~conditionally convergent~~ b/c $\frac{|x-\pi|^{N+1}}{(N+1)!} > 0$
☐ divergent

Ratio Test $\quad \rho = \lim_{N\to\infty} \frac{|x-\pi|^{N+2}}{(N+2)!} \frac{(N+1)!}{|x-\pi|^{N+1}} = |x-\pi| \lim_{N\to\infty} \frac{1}{N+2} = |x-\pi| \cdot 0 < 1$

So, since for all $x \in (-\infty, \infty)$

$$\lim_{N\to\infty} R_N(x) = 0,$$

we know that

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} (x-\pi)^{2n+1}$$

for all $x \in (-\infty, \infty)$.

Aside Useful fact. For all $x \in \mathbb{R}$:

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 = \lim_{n\to\infty} \left|\frac{x^n}{n!}\right| = \lim_{n\to\infty} \frac{|x|^n}{n!}$$

why? b/c even more is true, namely $\sum_n \frac{x^n}{n!}$ is abs. conv. by ratio test b/c $\rho = \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} \frac{n!}{|x|^n} = |x| \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1.$

Example 3 Let $x_0 = 0$ and

$$f(x) = \ln(1+x)$$

(2a) Find the Taylor series for $y = f(x)$ about $x_0 = 0$.

$$f(x) = \ln(1+x) \longrightarrow f^{0}(0) = f(0) = \ln 1 = 0$$

$$f^{(1)}(x) = (1+x)^{-1}$$

$$f^{(2)}(x) = -(1+x)^{-2}$$

$$f^{(3)}(x) = 2(1+x)^{-3}$$

$$f^{(4)}(x) = -2 \cdot 3(1+x)^{-4}$$

$$f^{(5)}(x) = 2 \cdot 3 \cdot 4(1+x)^{-5}$$

⋮ we see the pattern

need to compute $f^{(0)}(0)$ separately

$\boxed{n \geq 1}$

$\boxed{\text{for } n \geq 1} \longrightarrow f^{(n)}(x) = (-1)^{(n-1)}(n-1)!\,(1+x)^{-n} \Rightarrow f^{(n)}(0) = (-1)^{(n-1)}(n-1)!$

So
$$P_\infty(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n$$

$$= f(0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

↓ need $n \geq 1$ to use so start Σ at $n=$

$$= 0 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!} x^n$$

$$\Rightarrow \boxed{P_\infty(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n}$$

(2b) Remark. If we apply methods from power series (§ 10.6)

then we see that the interval of convergence

of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$ is $(-1, +1]$. ← (Ratio Test)

This you can do.

(2c) Fact

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n \quad \underbrace{\text{for } x \in (-1, +1]}_{\text{i.e. } -1 < x \le 1}.$$

But the proof of this Fact is very hard for $-1 < x < -\frac{1}{2}$.

So let's do an easier problem.

(2d) Show that for $-\frac{1}{4} \le x \le \frac{5}{8}$.

$$\underbrace{\ln(1+x)}_{f(x)} = \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n}_{\text{Taylor series for } f(x) \text{ about } x_0 = 0}.$$

Sol'n.

Here the N^{th} order Taylor polynomial is

$$P_N(x) = \sum_{n=1}^{N} \frac{(-1)^{n-1}}{n} x^n.$$

Know

$$f(x) = P_N(x) + R_N(x).$$

Fix $x \in \left[-\frac{1}{4}, \frac{5}{8}\right]$. We want to show

$$\lim_{N\to\infty} R_N(x) = 0$$

or equivalently

$$\lim_{N\to\infty} |R_N(x)| = 0.$$

$\boxed{\text{BIG Theorem}}$ for some c between x and $0 \leftarrow x_0$

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}(x-0)^{N+1}$$

$$\Rightarrow \quad |R_N(x)| = |f^{(N+1)}(c)| \, \frac{|x|^{N+1}}{(N+1)!}$$

ALREADY DID

$$f^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{(1+x)^n}$$

Let $n = N+1$

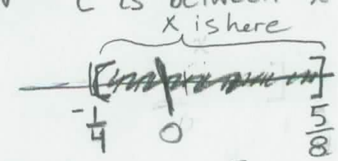
$x = c$

$$= \left| \frac{(-1)^N \, N!}{(1+c)^{N+1}} \right| \frac{|x|^{N+1}}{(N+1)!}$$

$$= \frac{1}{N+1} \cdot \left[\frac{|x|}{|1+c|}\right]^{N+1}$$

Given $-\frac{1}{4} \le x \le \frac{5}{8}$

Know c is between x & 0



So $-\frac{1}{4} \le c \le \frac{5}{8}$

$\rightarrow$ add 1 across

So $\frac{3}{4} \le 1+c \le \frac{13}{8}$

$$\le \frac{1}{N+1}\left[\frac{5/8}{3/4}\right]^{N+1}$$

$$= \frac{1}{N+1}\left(\frac{5}{6}\right)^{N+1}$$

$\boxed{\left(\frac{5}{6}\right)^{N+1} \le 1}$

$$\le \frac{1}{N+1} \xrightarrow{N\to\infty} 0$$

So $\lim_{N\to\infty} R_N(x) = 0$ if $-\frac{1}{4} \le x \le \frac{5}{8}$.

So $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$ if $-\frac{1}{4} \le x \le \frac{5}{8}$