§ 11.4 Comparison Test (CT) & Limit Comparison Test (LCT)

⇒ For positive termed series

⇕

$\sum_{n=17}^{\infty} a_n$ with $a_n > 0$

$\Big\Downarrow$ key idea

$$\sum_{n=17}^{\infty} a_n = \begin{cases} \bullet \text{ converges to a finite \#} \\ \boxed{\text{or}} \\ \bullet \text{ diverges to } \infty. \end{cases}$$

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(1) Comparison Test $\boxed{CT}$

- Book - page 705.

or

- class handout → read → why is it true?

Ex 1a $\sum_{n=2}^{\infty} \frac{1}{n^2+1}$ ☐ converge ☐ diverge

Well $0 < a_n \equiv \frac{1}{n^2+1} \overset{n \text{ big}}{\approx} \frac{1}{n^2}$

$\Rightarrow \sum a_n \approx \sum \frac{1}{n^2} < \infty$ ← p-series, $p = 2$, $p > 1$.

Guess $\sum_{n=2}^{\infty} \frac{1}{n^2+1}$ converges (to a finite #)

Let's use $\boxed{CT}$ to confirm our guess

Want $\sum \frac{1}{n^2+1} \underset{\uparrow}{\leq} \sum b_n \lneq \infty$

$a_n := \frac{1}{n^2+1} \leq b_n$

need to find b_n so that

① $\sum b_n \lneq \infty$

② $0 < a_n \leq b_n$

$$\boxed{0 < a_n = \frac{1}{n^2+1} \leq \frac{1}{n^2} = b_n}$$ ← we will come back to in Ex 2a.

$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n^2+1} \leq \sum_{n=2}^{\infty} \frac{1}{n^2} < \infty$

$\boxed{\text{p-series, } p = 2, \ p > 1 \text{ converges}}$

(CT) $\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n^2+1}$ converges

Ex 16 $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n+17}$ ☐ converges ☐ diverges

Well $0 < a_n \equiv \frac{1}{n+17} \overset{n \text{ big}}{\approx} \frac{1}{n}$

$\Rightarrow \sum \frac{1}{n+17} \approx \sum \frac{1}{n} = \infty$

harmonic series

or p-series, $p=1$, $p \le 1$

Guess $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n+17} = \infty$

Let's use $\boxed{CT}$ to confirm our guess.

Want $\infty = \sum b_n \le \sum \frac{1}{n+17}$

$b_n \le \frac{1}{n+17}$

need to find b_n so that

① $\sum b_n = \infty$

② $0 < b_n \le \frac{1}{n+17}$

$a_n \equiv \frac{1}{n+17} \ge \frac{1}{n+n} = \frac{1}{2n} \equiv b_n$

if $n \ge 17$

harmonic series

$\Rightarrow \displaystyle\sum_{n=17}^{\infty} \frac{1}{n+17} \ge \sum_{n=17}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=17}^{\infty} \frac{1}{n} = \infty$

$\overset{(CT)}{\Longrightarrow} \displaystyle\sum_{n=17}^{\infty} \frac{1}{n+17} = \infty$

$\Rightarrow \displaystyle\sum_{n=2}^{\infty} \frac{1}{n+17} = \infty$

$\Rightarrow \displaystyle\sum_{n=2}^{\infty} \frac{1}{n+17}$ diverges

(2) Limit Comparison Test $\boxed{LCT}$

- Book - page 707.

(or)

- class handout → read → why is it true?

Ex 2a $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ ☐ converges ☐ diverges

Comments on Ex 2a

① Compare Ex 2a with Ex 1a

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1} \qquad \sum_{n=2}^{\infty} \frac{1}{n^2+1} \leftarrow \text{he converges}$$

② So $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ probably converges.

Can show $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ converges by the $\boxed{CT}$ ....

would have to replace, in Ex 1a,

$$0 < a_n = \frac{1}{n^2+1} \leq \frac{1}{n^2} = b_n \quad \leftarrow \text{Ex 1a.}$$

by

$$0 < a_n = \frac{1}{n^2-1} \leq \frac{1}{n^2 - \frac{n^2}{2}} = \frac{1}{\frac{n^2}{2}} = \frac{2}{n^2} = b_n \leftarrow \text{Ex 2a}$$

hard-yek -

Remark For this reason, the $\boxed{LCT}$ is often easier than the $\boxed{CT}$.

Let's do Ex 2a using the LCT.

Ex 2a Continued

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}$$ ☐ converge ☐ diverge

Well $0 < a_n \equiv \frac{1}{n^2-1} \overset{n \text{ big}}{\sim} \frac{1}{n^2} = b_n$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_n}{1} \cdot \frac{1}{b_n}$$ ← how to think of — from now on I'll skip this step

$$= \lim_{n\to\infty} \frac{n^2}{n^2-1} \underset{\div \text{ thru by } n^2}{=} \lim_{n\to\infty} \frac{1}{1-\frac{1}{n^2}} = \frac{1}{1-0} = 1 \equiv L$$

$0 < 1 < \infty$

(LCT) Σa_n & Σb_n do the same thing

$\Sigma b_n = \Sigma \frac{1}{n^2}$ converges (p-series, $p=2$, $p>1$)

so $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ converges.

Ex 2b $$\sum_{n=1}^{\infty} \frac{n^{3/2}-5}{7n^2-8n-17}$$ ☐ converges ☐ diverges

$$\boxed{b_n = \frac{1}{n^{1/2}}}$$

⇑

Well $0 \overset{n \text{ big}}{<} a_n = \frac{n^{3/2}-5}{7n^2-8n-17} \overset{n \text{ big}}{\sim} \frac{n^{3/2}}{7n^2} = \frac{1}{7} \frac{1}{n^{1/2}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{3/2}-5}{7n^2-8n-17} \cdot \frac{n^{1/2}}{1} = \lim_{n\to\infty} \frac{n^2-5n^{1/2}}{7n^2-8n-17}$$ ← divide thru by n^2

$$= \lim_{n\to\infty} \frac{1-\frac{5}{n^{3/2}}}{7-\frac{8}{n}-\frac{17}{n^2}} = \frac{1-0}{7-0-0} = \frac{1}{7} = L$$

$0 < \frac{1}{7} < \infty$

(LCT) Σa_n & Σb_n do the same thing

$\Sigma b_n = \Sigma \frac{1}{n^{1/2}}$ diverges b/c p-series, $p=\frac{1}{2}$, $p \le 1$

so Σa_n diverges

Graph, for $x > 0$, on same grid

$f(x) = e^x \Rightarrow f'(x) = e^x$

$g(x) = \ln x . \Rightarrow g'(x) = \frac{1}{x}$

- Think about their "growth rates", ie, their derivatives.

- Now add to your graph grid: $y = x$, $y = x^2$, $y = x^{1/2}$, ...

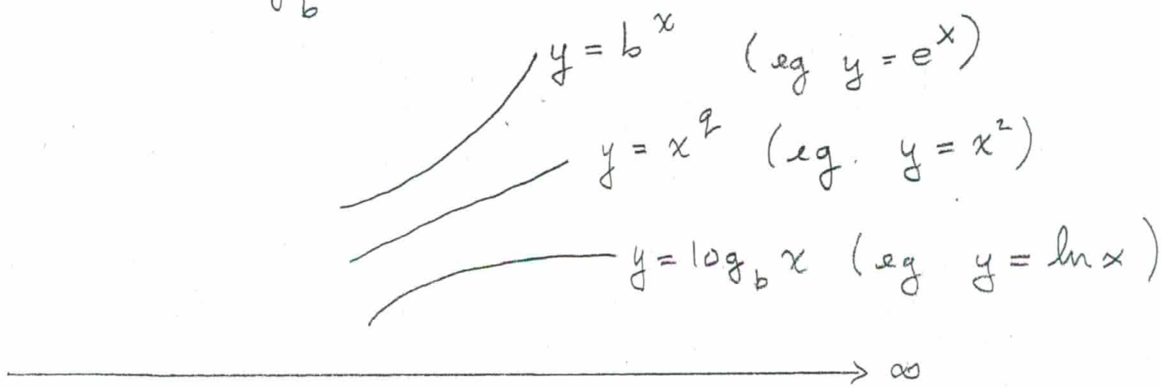
11.27 handout

Series Comparison Test

Helpful Intuition

For any power $0 < q < \infty$ and any base $b > 1$, for n large enough

$$\log_b n \le n^q \le b^n .$$



In fact (L'Hopital's Rule)

$$\lim_{n\to\infty} \frac{\log_b n}{n^q} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{n^q}{b^n} = 0.$$

So to figure out what happens to a series which involves a $\log_b n$ or b^n, remember

- $\log_b n$ grows "super slow" compared to n^q
- b^n grows "super fast" compared to n^q

An Example using helpful intuition.

$\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$ ☐ converges ☐ diverges

- $a_n = \frac{\ln n}{n^{3/2}} \underset{n \geq 2}{>} 0 \Rightarrow$ positive term series ← $\ln 1 = 0 \Rightarrow a_1 = 0$.

- Helpful Intuition $\ln n = \log_b n$ where $b \overset{?}{=} \boxed{e}$

bs $a_n = \frac{\ln n}{n^{3/2}}$

(1) $\ln n$ grows slow compared to n^q ← for any $0 < q < \infty$

(2) $\sum \frac{1}{n^{3/2}}$ converges (why? p-series, $p = 3/2 > 1$).

(3) so $\sum \frac{\ln n}{n^{3/2}}$ should converges. ← our guess

- Confirm our guess:

$0 \underset{n \geq 2}{<} a_n = \frac{\ln n}{n^{3/2}} \leq \frac{n^q}{n^{3/2}} \overset{\text{alg.}}{=} \frac{1}{n^{3/2-q}} \overset{q=\frac{1}{4}}{=} \frac{1}{n^{3/2-1/4}} = \frac{1}{n^{5/4}} = b_n.$

for any $0 < q < \infty$ as long as n is big enough

p-series $p = 3/2 - q$

want $3/2 - q > 1 \iff \frac{3}{2} - 1 > q \iff \frac{1}{2} > q > 0$

e.g. $q = 1/4$

$\sum b_n = \sum \frac{1}{n^{5/4}}$ conv., p-series, $p = \frac{5}{4} > 1$

$\underline{CT} \Rightarrow \sum_{n=1}^{\infty} a_n$ converges

11.29

Limit Comparison Test – beefed up.

see class handout on Series Summary

40. (a) Suppose that Σa_n and Σb_n are series with positive terms and Σb_n is convergent. Prove that if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0$$

then Σa_n is also convergent.

(b) Use part (a) to show that the series converges.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}\, e^n}$

$\boxed{41.}$ (a) Suppose that Σa_n and Σb_n are series with positive terms and Σb_n is divergent. Prove that if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$$

then Σa_n is also divergent.

(b) Use part (a) to show that the series diverges.

(i) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$ (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$

→ Prof G – explain to them!

→ Students – can also do 40 b & 41 b with "Helpful Intuition"
– it's part of your homework ☺.