# § 11.2 Infinite Series

Recall the Important Example from class lecture on §11.1.

**Ex 0** Fix a real number $r$ (so $r \in \mathbb{R}$).

Consider the sequence

$$\{r^N\}_{N=1}^{\infty} = \{r, r^2, r^3, r^4, r^5, \ldots\}.$$

Then

$$\lim_{N\to\infty} r^N = \begin{cases} 0 & \text{if } |r|<1 \\ 1 & \text{if } r=1 \\ \text{diverges (i.e. DNE)} & \text{if } |r|>1 \text{ or } r=-1. \end{cases}$$

**So.**

$$\lim_{N\to\infty} \left(\frac{1}{2}\right)^N = 0$$

$$\lim_{N\to\infty} \left(\frac{3}{2}\right)^N = \text{diverges (to } \infty)$$

$$\lim_{N\to\infty} \left(-\frac{3}{2}\right)^N = \text{diverges}$$

$\boxed{\text{Example 1 : against the wall}}$

I am standing 1 foot from the wall and am going to step towards the wall.
With each step, I go half the distance remaining to the wall.
How far do I go?

$$\frac{1}{2} + \frac{1}{2}\left[\text{of the distance remaining to the wall, which is } \frac{1}{2}\right]$$

$$+\frac{1}{2}\left[\text{of the distance remaining to the wall, which is } \left(\frac{1}{2}\right)\cdot\left(\frac{1}{2}\right)\right]$$

$$+\frac{1}{2}\left[\text{of the distance remaining to the wall, which is } \left(\frac{1}{2}\right)\cdot\left(\frac{1}{2}\right)\cdot\left(\frac{1}{2}\right)\right] + \cdots \quad \hookleftarrow \quad \text{the sum keeps on going}$$

$$= \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \cdots \quad \hookleftarrow \quad \text{the sum keeps on going}$$

$$= \left(\frac{1}{2}\right) + \left(\frac{1}{4}\right) + \left(\frac{1}{8}\right) + \left(\frac{1}{16}\right) + \cdots \quad \hookleftarrow \quad \text{the sum keeps on going}$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$$

now let $a_n = \left(\frac{1}{2}\right)^n$

$$= \sum_{n=1}^{\infty} a_n$$

$=$ ??? what ???

$=$ ok, we see the answer

$=$ but why , what do we REALLY mean by this <u>INFINITE SUM</u> ????

$$\overset{\text{well}}{=} \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \overset{\text{we can look at it this way}}{=} \lim_{N\to\infty} \left[\sum_{n=1}^{N} \left(\frac{1}{2}\right)^n\right]$$

$$\overset{\text{takes some algebra}}{=} \lim_{N\to\infty} \left[1 - \left(\frac{1}{2}\right)^N\right] \overset{\text{from our knowledge of sequences}}{=} 1 - 0 = 1$$

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10.13

## Ex 1

Let's do the algebra behind the "<u>takes some algebra</u>".

Show that

$$\sum_{n=1}^{N} \left(\tfrac{1}{2}\right)^n = 1 - \left(\tfrac{1}{2}\right)^N .$$

Let

$$S_N = \sum_{n=1}^{N} \left(\tfrac{1}{2}\right)^n .$$

Then

$$1 \quad S_N = \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^3 + \dots + \left(\tfrac{1}{2}\right)^N$$

trick

$$\tfrac{1}{2} \quad S_N = \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^3 + \dots + \left(\tfrac{1}{2}\right)^N + \left(\tfrac{1}{2}\right)^{N+1}$$

Subtract

$$\left(1 - \tfrac{1}{2}\right) S_N = \tfrac{1}{2} - \left(\tfrac{1}{2}\right)^{N+1}$$

$$\Rightarrow S_N = \frac{\frac{1}{2} - \left(\frac{1}{2}\right)^{N+1}}{1 - \frac{1}{2}} = \frac{\frac{1}{2} - \frac{1}{2}\left(\frac{1}{2}\right)^N}{\frac{1}{2}} = \frac{\frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^N\right)}{\frac{1}{2}}$$

$$\Rightarrow S_N = 1 - \left(\tfrac{1}{2}\right)^N$$

Start with an INFINITE SERIES $\sum_{n=1}^{\infty} a_n$. Think of it as

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_n + \ldots \quad \hookleftarrow \quad \text{the sum keeps on going and going - it is formal.}$$

Then form its corresponding sequence $\{s_N\}_{N=1}^{\infty}$ of partial sums where

$$s_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + a_3 + \ldots + a_N \quad \hookleftarrow \quad \text{the sum stops at } N.$$

Then

$$\sum_{n=1}^{\infty} a_n \text{ converges to } a \iff \lim_{N\to\infty} \sum_{n=1}^{N} a_n = a \iff \lim_{N\to\infty} s_N = a$$

and

$$\sum_{n=1}^{\infty} a_n \text{ diverges} \iff \lim_{N\to\infty} \sum_{n=1}^{N} a_n \text{ DNE} \iff \lim_{N\to\infty} s_N \text{ DNE.}$$

Rmk: $k=1$, doesn't matter where start

## Algebraic Properties of Series

Let $\Sigma a_n$ and $\Sigma b_n$ be convergent series. Let $c$ be a real number.
Then the following series converge: $\sum (ca_n)$ and $\sum (a_n+b_n)$ and $\sum (a_n - b_n)$.
Futhermore, they converge to what they should converge to, i.e.

$$\sum (ca_n) = c\left(\sum a_n\right) \quad \text{and} \quad \sum (a_n+b_n) = \left(\Sigma a_n\right) + \left(\Sigma b_n\right) \quad \text{and} \quad \sum (a_n - b_n) = \left(\Sigma a_n\right) - \left(\Sigma b_n\right)$$

kinda like a distributive property

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Proof that if $\sum_{n=1}^{\infty} a_n$ converges and $c \in \mathbb{R}$ then $\sum_{n=1}^{\infty} (ca_n)$ converges and $\sum_{n=1}^{\infty} (ca_n) = c \sum_{n=1}^{\infty} a_n$.

Let $c \in \mathbb{R}$ and $\sum_{n=1}^{\infty} a_n$ converges, say to $L$.

Then $L \overset{(1)}{:=} \sum_{n=1}^{\infty} a_n \;\underset{=}{\text{def}}\; \lim_{N\to\infty} \sum_{n=1}^{N} a_n$.

Note that for each $N \in \mathbb{N} := \{1, 2, 3, \dots\}$

$$\sum_{n=1}^{N} (ca_n) = c \sum_{n=1}^{N} a_n . \qquad (2)$$

So

$$\lim_{N\to\infty} \sum_{n=1}^{N} (ca_n) \overset{(2)}{=} \lim_{N\to\infty} \left( c \sum_{n=1}^{N} a_n \right)$$

$$= c \left( \lim_{N\to\infty} \sum_{n=1}^{N} a_n \right)$$

$$\overset{(1)}{=} c \cdot L$$

Thus $\sum_{n=1}^{\infty} (ca_n)$ converges and

$$\sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n . \qquad \blacksquare$$

Proof of the other 2 algebraic properties – for you.

<u>Theorem</u> "It doesn't matter where you start"

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \text{for } \underline{\underline{\text{each}}} \ K \in \mathbb{N},\ \sum_{n=K}^{\infty} a_n \text{ converges.}$$

<u>Why?</u>

$$\sum_{n=1}^{\infty} a_n \overset{\text{def}}{=\!=} \lim_{N\to\infty} \sum_{n=1}^{N} a_n$$

$$= \lim_{N\to\infty} \left[ \sum_{n=1}^{K-1} a_n + \sum_{n=K}^{N} a_n \right]$$

$$= \left[ \lim_{N\to\infty} \underbrace{\sum_{n=1}^{K-1} a_n}_{\text{constant wrt } N} \right] + \underbrace{\left[ \lim_{N\to\infty} \sum_{n=K}^{N} a_n \right]}_{\text{def}}$$

$$= \sum_{n=1}^{K-1} a_n + \sum_{n=K}^{\infty} a_n$$

$\blacksquare$

<u>Theorem</u> $n^{\text{th}}$ term test <u>for divergence</u>

<u>Logic</u> $\displaystyle\sum_{n=1}^{\infty} a_n$ conv. $\Rightarrow \displaystyle\lim_{n\to\infty} a_n = 0.$

<u>Working Form</u> $\displaystyle\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ divg.

Exercise 22 $\quad \displaystyle\sum_{n=17}^{\infty} \frac{n+1}{2n-3}$ $\quad$ ☐ converges $\quad$ ☒ diverges.

$$a_n = \frac{n+1}{2n-3} \stackrel{\text{i.e.}}{=} \frac{1n+1}{2n-3} \xrightarrow{n\to\infty} \frac{1}{2} \neq 0.$$

So $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{2n-3}$ divg. by $n^{\text{th}}$ term test for divergence.

So $\displaystyle\sum_{n=17}^{\infty} \frac{n+1}{2n-3}$ divg. by ________________ theorem.

## Remark

If $\displaystyle\lim_{n\to\infty} a_n = 0$, then know nothing about $\displaystyle\sum_{n=1}^{\infty} a_n$.

For example:

500-level math

(1) $\displaystyle\lim_{n\to\infty} \frac{1}{n^2} = 0$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

(2) $\displaystyle\lim_{n\to\infty} \frac{1}{n} = 0$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = \infty$

book proves in §11.2.
we will prove in §11.3.
can use on hmwk in 11.2.

The key idea behind all of this is
how fast does $a_n$ go to zero.

we will develop "test" to measure how fast is fast enough.

# Geometric Series, with ratio $r$ (and $c \neq 0$).

$$\sum_{n=0}^{\infty} cr^n = c(1+r+r^2+r^3+\dots) = \begin{cases} \text{converge} & \text{if } |r|<1 \\ \text{diverge} & \text{if } |r| \geq 1 \end{cases}$$

Proof. (you need to understand in order to do the problems)

By Algebraic Properties for series, it is enough to show that:

$$\sum_{n=0}^{\infty} r^n = \begin{cases} \text{converges} & \text{if } |r|<1 \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases} \qquad (*)$$

Let

$$S_N = \sum_{n=0}^{N} r^n .$$

So

?

$$S_N = 1 + r^1 + r^2 + r^3 + \dots + r^N$$

$$(r)\, S_N = \quad r^1 + r^2 + r^3 + \dots + r^N + r^{N+1}$$

subtract

$$(1-r)\, S_N = 1 - r^{N+1}$$

Case $r \neq 1$ ← so $1-r \neq 0$ so we can divide by $1-r$

$$\sum_{n=0}^{\infty} r^n \overset{\text{def}}{=} \lim_{N\to\infty} \sum_{n=0}^{N} r^n = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{1-r^{N+1}}{1-r} = \frac{1}{1-r} \lim_{N\to\infty} \left[1-r^{N+1}\right]$$

$$= \frac{1}{1-r}\left[1 - \lim_{N\to\infty} r^{N+1}\right] \overset{\text{Ex0}}{=} \begin{cases} \text{converge} & \text{if } |r|<1 \\ \text{diverge} & \text{if } |r|>1 \text{ or } r=-1. \end{cases}$$

Case $r = 1$

$$S_N = 1 + \overbrace{1^1 + 1^2 + \dots + 1^N}^{N\text{-term}} = N+1 \xrightarrow{N\to\infty} \infty$$

$$\Rightarrow \sum_{n=0}^{\infty} (1)^n = \infty \text{ or diverges or DNE}.$$

Latin for "all done"

Q.E.D.

Rmk. See Infinite Series - Summary handout - Geometric Series part

# How to detect a Geometric Series

Claim $\sum_{n=0}^{\infty} a_n$ is a Geometric Series with ratio $r$ $\iff$ $\frac{a_{n+1}}{a_n} = r \quad \forall n \geq 0$.

$\boxed{\Rightarrow}$ $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} c r^n \Rightarrow \frac{a_{n+1}}{a_n} = \frac{c r^{n+1}}{c r^n} = r.$

$\boxed{\Leftarrow}$ Let $\forall n \geq 0$ $\frac{a_{n+1}}{a_n} = r \Rightarrow a_{n+1} \overset{*}{=} (a_n) r$. So

Let's call $a_0 = c$.

$a_1 \overset{*}{=} (a_0) r = c r$

$a_2 \overset{*}{=} (a_1) r = (cr) r = c r^2$

$a_3 \overset{*}{=} (a_2) r = (c r^2) r = c r^3$ got the picture?

Ex 2 Show that $\sum_{n=17}^{\infty} 5 \frac{(-2)^n}{3^{2n+1}}$ is a geo. series

$a_n = 5 \frac{(-2)^n}{3^{2n+1}}$

$\Rightarrow \frac{a_{n+1}}{a_n} = \frac{\cancel{5} \frac{(-2)^{n+1}}{3^{2(n+1)+1}}}{\cancel{5} \frac{(-2)^n}{3^{2n+1}}} = \frac{\frac{(-2)^{n+1}}{3^{2n+3}}}{\frac{(-2)^n}{3^{2n+1}}}$ be careful

$= \frac{(-2)^{n+1}}{3^{2n+3}} \cdot \frac{3^{2n+1}}{(-2)^n} = \frac{(-2)^{n+1}}{(-2)^n} \cdot \frac{3^{2n+1}}{3^{2n+3}}$

$= (-2)^1 \cdot \frac{1}{3^2} = \boxed{-\frac{2}{9}}$

Ex 3 Consider $\sum_{n=17}^{\infty} 5 \frac{(-2)^n}{3^{2n+1}}$. Let $S_N = \sum_{n=17}^{N} 5 \frac{(-2)^n}{3^{2n+1}}$.

① Express $S_N$ w/out a $\Sigma$ sign

② Determine if the infinite series conv or div.

③ If it conv., find its sum.

- Ex 2 $\Rightarrow$ geo. series w/ $r = -\frac{2}{9} \Rightarrow |r|<1 \Rightarrow$ conv.
- Now to find $S_N$ .... time for "alg. clean up"

Write $5 \frac{(-2)^n}{3^{2n+1}} \stackrel{as}{=} c\, r^n$ ....

$$5 \frac{(-2)^n}{3^{2n+1}} = 5 \frac{(-2)^n}{3^{2n}\cdot 3} = \frac{5}{3} \frac{(-2)^n}{3^{2n}} = \frac{5}{3} \frac{(-2)^n}{(3^2)^n} = \frac{5}{3}\left(\frac{-2}{9}\right)^n .$$

- Do the easy tick ..

?

$$S_N = \frac{5}{3}\left[ \left(\frac{-2}{9}\right)^{17} + \left(\frac{-2}{9}\right)^{18} + \left(-\frac{2}{5}\right)^{19} + \cdots + \left(-\frac{2}{5}\right)^{N-1} + \left(\frac{-2}{5}\right)^{N} \right]$$

$$\left(\frac{-2}{9}\right) S_N = \frac{5}{3}\left[ \quad (\ )^{18} + (\ )^{19} + \cdots + (\ )^{N} + (\ )^{N+1} \right]$$

subtract

$$\underbrace{\left(1 - \left(\frac{-2}{9}\right)\right)}_{\frac{11}{9}} S_N = \frac{5}{3}\left[ \left(-\frac{2}{9}\right)^{17} - \left(-\frac{2}{9}\right)^{N+1} \right]$$

$$\boxed{S_N = \frac{9}{11}\cdot\frac{5}{3}\left[ \left(-\frac{2}{9}\right)^{17} - \left(-\frac{2}{9}\right)^{N+1} \right]} \xrightarrow{N\to\infty} \boxed{\frac{9}{11}\cdot\frac{5}{3}\left[ \left(\frac{-2}{9}\right)^{17} - 0 \right]}$$

$$\| \quad \sum_{n=17}^{\infty} 5 \frac{(-2)^n}{3^{2n+1}}$$

$$1 - \frac{-2}{9} = \frac{9}{9} + \frac{2}{9} = \frac{11}{9}.$$

ps - an exam, this is "far enough"

Ex 4 $\quad \sum_{k=1}^{\infty} \frac{2}{k(k+2)}$ ← Not a geometric series but it is a Telescoping Series

$$\frac{2}{k(k+2)} \underset{=}{\overset{PFD}{=}} \frac{A}{k} + \frac{B}{k+2} \xrightarrow{\text{work}} \frac{1}{k} + \frac{-1}{k+2}$$

Look at partial sums $S_N$ :

$$S_N = \sum_{k=1}^{N} \frac{2}{k(k+2)} = \sum_{k=1}^{N} \left[ \frac{1}{k} + \frac{-1}{k+2} \right]$$

$$\left.\begin{aligned}
&= 1 + -\tfrac{1}{3} && \leftarrow k=1 \\
&\quad + \tfrac{1}{2} + \tfrac{-1}{4} && \leftarrow k=2 \\
&\quad + \tfrac{1}{3} + -\tfrac{1}{5} && \leftarrow k=3 \\
&\quad + \tfrac{1}{4} + -\tfrac{1}{6} && \leftarrow k=4 \\
&\quad \vdots \\
&\quad + \tfrac{1}{N} + \tfrac{-1}{N+2} && \leftarrow k=N
\end{aligned}\right\}$$

$k$ & $k+2$ differ by $\underline{\underline{2}}$ so regroup in sets of 2's

Ex 4 — continued

$$S_N = 1 + \frac{-1}{3} + \frac{1}{2} + \frac{-1}{4} \quad \leftarrow k = 1, 2$$

$$+ \frac{1}{3} + \frac{-1}{5} + \frac{1}{4} + \frac{-1}{6} \quad \leftarrow k = 3, 4$$

$$+ \frac{1}{5} + \frac{-1}{7} + \frac{1}{6} + \frac{-1}{8} \quad \leftarrow k = 5, 6$$

$$+ \frac{1}{7} + \frac{-1}{9} + \frac{1}{8} + \frac{-1}{10} \quad \leftarrow k = 7, 8$$

$$\vdots$$

$$+ \frac{1}{N-1} + \frac{-1}{N+1} + \frac{1}{N} + \frac{-1}{N+2} \quad \leftarrow k = N-1, N$$

$$\boxed{S_N = 1 + \frac{1}{2} + \frac{-1}{N+1} + \frac{-1}{N+2}} \xrightarrow{N \to \infty} 1 + \frac{1}{2} + 0 + 0 = \frac{3}{2}.$$

So $\sum_{k=1}^{\infty} \frac{2}{k(k+2)}$ converges and $\sum_{k=1}^{\infty} \frac{2}{k(k+2)} = \boxed{\frac{3}{2}}$

Self Check

$$S_3 = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) = \frac{3}{2} - \frac{1}{3+1} + \frac{-1}{3+2}$$

$$S_4 = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} + \frac{-1}{6}\right)$$

$$= \frac{3}{2} + \frac{-1}{4+1} + \frac{-1}{4+2}$$

etc...