§ 11.?? : Between 11.4 and 11.5

① Recall, so far we had: $\sum a_n$ ☐ converge ☐ diverge

if $a_n \geq 0$, can use: Integral Test, CT, LCT

② Now we consider series $\sum a_n$ where

- some of the a_n's might be $a_n \geq 0$
- some of the a_n's might be $a_n \leq 0$.

③ Now our choices are $\sum_{n=1}^{\infty} a_n$ ☐ absolutely convergent ☐ conditionally convergent ☐ divergent

old way $a_n \geq 0$: $\sum a_n$ ☐ conv. ☐ divg.

conv. → ☐ absolutely convergent ☐ conditionally convergent

now we divide into 2 categories

④ Let's look at Infinite Series Summary handout:

Def's, Big Theorem (read proof in book), &

Mutually Exclusive and Exhaustive Possibilities

§ 11.5 Alternating Series Test

Def. Alternating Series (notation differs from book but is consist w/ my handout)

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n u_n \quad \text{and} \quad u_n > 0$$

$n=1$ or 17

$$\text{So } \sum_{n=1}^{\infty} (-1)^n u_n = \underbrace{-u_1 + u_2 - u_3 + u_4 - u_5 + \cdots}_{u_n > 0 \Rightarrow \text{ alternating } +/- \text{ signs.}}$$

Alternating Series Test (AST) - see Series Summary Handout

Consider an alternating series $\sum (-1)^n u_n$ where $u_n > 0$

If: ① $u_n > u_{n+1} \overset{\text{free}}{>} 0 \quad \forall n \in \mathbb{N}$ (i.e. u_n's are decreasing)

② $\lim\limits_{n\to\infty} u_n = 0$

Then $\sum (-1)^n u_n$ converges.

Ex 1 $\quad \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

☐ absolutely convergent
☐ conditionally convergent
☐ divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n} = \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n u_n,$$

where $a_n = \dfrac{(-1)^n}{n}$ and $u_n = \dfrac{1}{n} > 0$.

① abs. conv.? $\rightarrow$ ☐

Consider $\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$

harmonic series
=or=
p-series w/
$p = 1 \le 1$

$\Rightarrow$ diverges.

② cond. conv? $\rightarrow$ ☐

$\sum \frac{(-1)^n}{n} = \sum (-1)^n u_n$ with $u_n = \frac{1}{n} > 0 \Rightarrow$ It's an alternating series.

AST ① $\lim\limits_{n\to\infty} u_n = 0$ ✓

② $u_{n+1} < u_n \overset{\text{i.e}}{\Longleftrightarrow} \frac{1}{n+1} < \frac{1}{n}$ ✓

AST $\Rightarrow \sum (-1)^n \frac{1}{n}$ converges.

③ So $\sum \left|\frac{(-1)^n}{n}\right|$ diverges
$\sum \frac{(-1)^n}{n}$ converges
$\Rightarrow \sum \frac{(-1)^n}{n}$ cond. converges

Ex 2. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+1}$ — ☐ abs. conv. ☐ cond. conv. ☐ divg.

Well This is really a beefed-up (made harder) version of Ex 1. Let's learn from (i.e. compare to) Ex 1.

Ex. 1. $\sum (-1)^n \frac{1}{n} = \sum (-1)^n \tilde{u}_n$

Ex 2 $\sum (-1)^n \frac{n^3}{n^4+1} = \sum (-1)^n u_n$

$$u_n = \frac{n^3}{n^4+1} \overset{n \text{ big}}{\approx} \frac{n^3}{n^4} = \frac{1}{n} = \tilde{u}_n$$

So $\underbrace{\sum (-1)^n u_n}$ should behave like $\sum (-1)^n \tilde{u}_n$.

$\Downarrow$ Ex 1

cond. conv.

he should also be cond. conv. $\Longleftarrow$

Now. to justify our above guess.

① abs. conv.? $\longrightarrow$ ☐

Consider $\sum \left| (-1)^n \frac{n^3}{n^4+1} \right| = \sum \frac{n^3}{n^4+1}$

$$0 < \frac{n^3}{n^4+1} \overset{n \text{ big}}{\approx} \frac{n^3}{n^4} = \frac{1}{n} = b_n$$

LCT. $\displaystyle\lim_{n\to\infty} \frac{\frac{n^3}{n^4+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^4}{n^4+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^4}} = \frac{1}{1+0} = 1$

$\downarrow$

$0 < 1 < \infty$.

So $\sum \frac{n^3}{n^4+1}$ & $\sum \frac{1}{n}$ "do the same thing." $\longrightarrow$

$\uparrow$ diverges p-series $p = 1 \leq 1$

So $\sum \frac{n^3}{n^4+1}$ diverges

(2) cond. conv.? $\longrightarrow$ $\boxed{}$

Consider $\sum (-1)^n \frac{n^3}{n^4+1} = \sum (-1)^n u_n$ with $u_n = \frac{n^3}{n^4+1} > 0.$

So it's an alternating series

AST. (1) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n^3}{n^4+1} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n^4}} = \frac{0}{1+0} = 0$ ✓

(2) Are the u_n's dearcasing?

$f(x) = \frac{x^3}{x^4+1}$ ← want dec. so want $f'(x) < 0$ for $x \geq 1$ (or 17 is good enough)

$f'(x) \underset{\text{work}}{\overset{\text{Calc I}}{=}} \frac{x^2(3-x^4)}{(x^4+1)^2} < 0 \iff 3 - x^4 < 0$

$\iff 3 < x^4$

so dec. for n big enough. $\iff 3^{1/4} < |x|$

≈ 1.3

AST. $\Rightarrow \sum \frac{(-1)^n n^3}{n^4+1}$ conv.

Conclusion

$\sum \left| \frac{(-1)^n n^3}{n^4+1} \right|$ divg

$\sum \frac{(-1)^n n^3}{n^4+1}$ convg.

$\Rightarrow \sum (-1)^n \frac{n^3}{n^4+1}$ is conditionally convergent.